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Introduction to the Mathematics of Finance

金融数学引论

R. J. Williams

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出版者的话

近年来，我国的科学技术取得了长足进步，特别是在数学等自然科学基础领域不断涌现出一流的研究成果。与此同时，国内的科研队伍与国外的交流合作也越来越密切，越来越多的科研工作者可以熟练地阅读英文文献，并在国际顶级期刊发表英文学术文章，在国外出版社出版英文学术著作。

然而，在国内阅读海外原版英文图书仍不是非常便捷。一方面，这些原版图书主要集中在科技、教育比较发达的大中城市的大型综合图书馆以及科研院所的资料室中，普通读者借阅不甚容易；另一方面，原版书价格昂贵，动辄上百美元，购买也很不方便。这极大地限制了科技工作者对于国外先进科学技术知识的获取，间接阻碍了我国科技的发展。

高等教育出版社本着植根教育、弘扬学术的宗旨服务我国广大科技和教育工作者，同美国数学会（American Mathematical Society）合作，在征求海内外众多专家学者意见的基础上，精选该学会近年出版的数十种专业著作，组织出版了“美国数学会经典影印系列”丛书。美国数学会创建于 1888 年，是国际上极具影响力的专业学术组织，目前拥有近 30000 会员和 580 余个机构成员，出版图书 3500 多种，冯·诺依曼、莱夫谢茨、陶哲轩等世界级数学大家都是其作者。本影印系列涵盖了代数、几何、分析、方程、拓扑、概率、动力系统等所有主要数学分支以及新近发展的数学主题。

我们希望这套书的出版，能够对国内的科研工作者、教育工作者以及青年学生起到重要的学术引领作用，也希望今后能有更多的海外优秀英文著作被介绍到中国。

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Preface

This monograph is intended as an introduction to some elements of mathematical finance. It begins with the development of the basic ideas of hedging and pricing of European and American derivatives in the discrete (i.e., discrete time and discrete state) setting of binomial tree models. Then a general discrete finite market model is defined and the fundamental theorems of asset pricing are proved in this setting. Tools from probability such as conditional expectation, filtration, (super)martingale, equivalent martingale measure, and martingale representation are all used first in this simple discrete framework. This is intended to provide a bridge to the continuous (time and state) setting which requires the additional concepts of Brownian motion and stochastic calculus. The simplest model in the continuous setting is the Black-Scholes model. For this, pricing and hedging of European and American derivatives are developed. The book concludes with a description of the fundamental theorems of asset pricing for a continuous market model that generalizes the simple Black-Scholes model in several directions.

The modern subject of mathematical finance has undergone considerable development, both in theory and practice, since the seminal work of Black and Scholes appeared a third of a century ago. The material presented here is intended to provide students and researchers with an introduction that will enable them to go on to read more advanced texts and research papers. Examples of topics for such further study include incomplete markets, interest rate models and credit derivatives.

For reading this book, a basic knowledge of probability theory at the level of the book by Chung [**10**] or D. Williams [**38**], plus for the chapters on continuous models, an acquaintance with stochastic calculus at the level of the book by Chung and Williams [**11**] or Karatzas and Shreve [**27**], is

desirable. To assist the reader in reviewing this material, a summary of some of the key concepts and results relating to conditional expectation, martingales, discrete and continuous time stochastic processes, Brownian motion and stochastic calculus is provided in the appendices. In particular, the basic theory of continuous time martingales and stochastic calculus for Brownian motion should be briefly reviewed before commencing Chapter 4. Appendices C and D may be used for this purpose.

Most of the results in the main body of the book are proved in detail. Notable exceptions are results from linear programming used in Section 3.5, results used for pricing American contingent claims based on continuous models in Sections 4.7 through 4.9, and several results related to the fundamental theorems of asset pricing for the multi-dimensional Black-Scholes model treated in Chapter 5.

I benefited from reading treatments of various topics in other books on mathematical finance, including those by Pliska [**33**], Lamberton and Lapeyre [**30**], Elliott and Kopp [**15**], Musiela and Rutkowski [**32**], Bingham and Kiesel [**4**], and Karatzas and Shreve [**28**], although the treatment presented here does not parallel any one of them.

This monograph is based on lectures I gave in a graduate course at the University of California, San Diego. The material in Chapters 1–3 was also used in adapted form for part of a junior/senior-level undergraduate course on discrete models in mathematical finance at UCSD. The students in both courses came principally from mathematics and economics. I would like to thank the students in the graduate course for taking notes which formed the starting point for this monograph. Special thanks go to Nick Loehr and Amber Puha for assistance in preparing and reading the notes, and to Judy Gregg and Zelinda Collins for technical typing of some parts of the manuscript. Thanks also go to Steven Bell, Sumit Bhardwaj, Jonathan Goodman and Raphael de Santiago, for providing helpful comments on versions of the notes. Finally, I thank Bill Helton for his continuous encouragement and good humor.

R. J. Williams
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Contents

Chapter 1

Financial Markets and Derivatives

1.1. Financial Markets

A (primary) *financial market* consists of tradable securities such as stocks, bonds, currencies, commodities, or even indexes. One reason for the existence of financial markets is that they facilitate the flow of capital. For example, if a company wants to finance the building of a new production facility, it might sell shares of stock to investors, who buy the shares based on the anticipation of future rewards such as dividends or a rise in the stock price.

A variety of stochastic models is used in modeling the prices of securities. All such models face the usual trade off: more complex models typically provide a better fit to data (although there is the danger of overfitting), whereas simpler models are generally more tractable and despite their simplicity can sometimes provide useful qualitative insights. Finding a good balance between a realistic and a tractable model is part of the art of stochastic modeling.

Both discrete (in time and state), and continuous (in time and state) models will be considered here. The treatment of discrete time models is intended as a means of introducing notions such as hedging and pricing by arbitrage in a simple setting and to provide a bridge to the development in the continuous case. Binomial tree models are the most common example in the discrete case. Our treatment of discrete and continuous models is not meant to be exhaustive, but rather to provide an introduction that will enable students to go on to read more advanced material.

An *arbitrage opportunity* is an opportunity for a risk free profit. A financial market is said to be *viable* if there are no arbitrage opportunities. Typically, liquid financial markets move rapidly to eliminate arbitrage opportunities. Accordingly, in this book, attention will focus on financial market models that are viable.

1.2. Derivatives

A *derivative* or *contingent claim* is a security whose value depends on the value of some underlying security. Examples of derivatives are forward contracts, futures contracts, options, swaps, etc. (We recognize that the reader may not be familiar with all of these terms. Some of them are described in more detail below. For the other terms, or more detailed descriptions in general, we refer the interested reader to the book by Hull [**23**] or Musiela and Rutkowski [**32**]. For an in-depth, non-mathematical description of market microstructure, see Harris [**19**].) The underlying security on which a derivative is based could be a security in a financial market, such as a stock, bond, currency or commodity, but it could also be a derivative itself, such as a futures contract.

Secondary financial markets can be formed from derivatives. Some derivatives are sold on public exchanges such as the Chicago Board of Options Exchange (CBOE) the American Stock and Options Exchange (AMEX), the Chicago Board of Trade (CBOT) and the Chicago Mercantile Exchange (CME). However, many derivatives are sold over-the-counter (OTC); i.e., they are traded between individual entities — typically financial institutions and/or their corporate clients. The "derivatives market" is huge, and the OTC market is substantially bigger than the exchange traded market. An indication of the size of the OTC market is provided in reports of the Bank for International Settlements. For example, a May 2005 report [**2**] states that the amount of principal outstanding in the global over-the-counter derivatives market at the end of December 2004 was $248 trillion.

We shall now describe some examples of derivatives in a bit more detail.

A *forward contract* is an agreement to buy or sell an asset at a certain future time for a certain price. Forward contracts are traded *over the counter* (OTC). One of the common uses of forward contracts is to lock in an exchange rate for a future purchase in a foreign currency.

A *futures contract* is similar to a forward contract, but it is traded on a financial exchange. Futures contracts (or simply futures) differ from forward contracts in a number of respects. In particular, futures typically have a delivery month rather than a delivery date, and they follow a settlement procedure called *marking to market.* Briefly, in this procedure, an investor's initial deposit is adjusted on each trading day to reflect gains or losses in

the futures price for that day. For further details, we refer the reader to [**23**] or [**32**]. Futures contracts are available for a wide range of commodities and financial assets. Exchanges on which futures are traded include the Chicago Board of Trade and the Chicago Mercantile Exchange.

An *option* is a contract which gives the holder of the option the right, but not the obligation, to buy or sell a given security at a given price (called the exercise price or *strike price*) within a fixed time period $[0, T]$. A *call option* gives the option holder the right to buy at the given price, whereas a *put option* gives the option holder the right to sell at the given price. A *European option* can only be exercised by the holder of the option at the expiration time T, whereas an *American option* can be exercised by the holder at any time in $[0, T]$. Exchanges on which options are traded include the Chicago Board of Options and the American Stock and Options Exchange.

Remark. Options based on stocks are usually written to cover the buying or selling of 100 shares of a stock. This is convenient, since shares of stock are usually traded in lots of 100. Most exchange traded options on stock are American-style options. Stock options on the Chicago Board Options Exchange effectively expire on the third Friday of the expiration month.

One might ask, why are options worth buying or selling? To help answer this, consider an example. To keep the description simple, consider a European call option.

Example. On January 4, 2000, a European call option on Cisco (symbol: CSCO) stock has a price of \$33. The option expires in January, and the strike price is \$70. The price of Cisco stock on January 4 is \$102.

If one bought such an option on 100 shares of Cisco, the option would cost \$3,300, and on January 21, 2000 (third Friday of January), one would have the right to buy 100 shares of Cisco at a price of \$70 per share. Suppose for simplicity that \$1 on January 4 is worth \$1 on January 21, 2000.

Scenario 1: Suppose the price of Cisco stock on January 21 is \$120 per share. This current price of the stock is called the *spot price* of the stock. The holder of the option will exercise it and make a net profit per share of $\$120 - \$70 - \$33$ (spot price of stock on January 21 − price under exercise of option − option price) and hence a net profit of $\$1,700$. This is a $\frac{1700}{33}\% = 51.5\%$ profit on the $\$3,300$ initial investment. On the other hand, if the $\$3,300$ had been directly invested in stock, the investor could have bought 32 whole shares of stock and the profit would have been $\$18 \times 32 = \$576$ on an investment of $\$102 \times 32 = \$3,264$, which is a $\frac{57600}{3264}\% = 17.6\%$ profit.

Scenario 2: Suppose the price of Cisco stock on January 21 is \$67 per share. The holder of the option will not exercise it and takes a loss of \$33

per share (the cost of the option per share) and hence a net loss of \$3,300. This is a 100% loss on the \$3,300 initial investment. On the other hand, if the \$3,300 had been invested directly in stock, the loss would have been $\$35 \times 32 = \$1,120$ or a 34.3% loss on an investment of \$3,264 in stock.

There are two main uses of derivatives, namely, *speculation and hedging*. For example, the buyer of a call option on a stock is leveraging his/her investment. The option may not cost much compared to the underlying stock, and the owner of the call option can benefit from a rise in the stock price without having to buy the stock. Of course, if the stock price goes down substantially and the option is not exercised, the buyer loses what he/she paid for the call option. In this case, the seller of the call option would benefit without necessarily having to buy the stock. Of course, if the stock price goes up substantially and the option seller does not already own the stock, then he/she will have to pay the price of buying the stock at the increased price and turning it over to the owner of the option. Options may be used to leverage and speculate in the market. Depending on how the investment is used, the downside losses can be substantial. On the other hand, options can be used to hedge risk. An investor *hedges* when he or she reduces the risk associated with his or her portfolio holdings by trading in the market to reduce exposure of the portfolio to the effect of random fluctuations in market prices. For example, an owner of stock who wants to hedge against a dramatic decrease in the stock price could buy a put option to sell the stock at a certain price, thereby reducing the risk of loss associated with such a dramatic decrease.

The focus of this book will be on the *pricing of derivative securities*. Derivatives will be referred to henceforth as contingent claims. Both European and American contingent claims will be considered here. (In this book, we restrict to contingent claims having a single payoff, cf. [**15**], rather than allowing the more general situation of [**28**], where a contingent claim can have an income stream over an interval of time.) A *European contingent claim* has a single expiration date that is the same as its payoff date. The owner of an *American contingent claim* can choose to cash in the claim at any time up until the expiration date, and the payoff may depend on when the owner chooses to cash in the claim. The main principle behind pricing of both types of contingent claims is that the price should be set in such a way that there is no opportunity for arbitrage in a market where the contingent claim and underlying securities are traded. The existence of an arbitrage free price for a contingent claim hinges on the assumption that there are no opportunities for arbitrage in the underlying (primary) securities market; i.e., that market is viable. Uniqueness of the arbitrage free price depends in the case of a European contingent claim on being able to replicate the payoff of the claim by trading in the (primary) securities market, and in the

case of an American contingent claim on the seller of the claim being able to hedge the sale by trading in the market so as to have a portfolio whose value equals or exceeds the payoff of the claim at all times and equals it at some (random) time. These hypotheses will be defined precisely in the context of a model later on, and conditions under which they hold will be given.

The following further assumptions are made to simplify the presentation. There are (a) no transaction costs and (b) no dividends on stock. It is assumed that (c) the market is liquid and (d) trading by the investor does not move the market. In the discrete model framework, an investor is allowed (e) unlimited short selling of stock and (f) unlimited borrowing. In the continuous model framework, these assumptions will be modified to prevent "doubling strategies" which result in an arbitrage opportunity.

1.3. Exercise

1. Using a list of option prices (e.g., from the website for the Chicago Board Options Exchange or Yahoo Finance), perform a similar calculation to that done in the Example in the text, with Intel (symbol: INTC) in place of Cisco. For this, use the first Intel call option expiring in the current month and for which a price is listed. You should assume that a European call option for 100 shares of stock is purchased and that at expiration there are two possible scenarios for the stock price: it has gone up or down by 40% since purchase of the option.

Chapter 2

Binomial Model

In this chapter we consider a simple discrete (primary) financial market model called the binomial or Cox-Ross-Rubinstein (CRR) [**12**] model. In the context of this model, we derive the unique arbitrage free prices for a European and an American contingent claim.

For this chapter and the next, we shall need various notions from probability and the theory of discrete time stochastic processes, including conditional expectations, martingales, supermartingales and stopping times. For the convenience of the reader, a summary of some relevant concepts and results is provided in Appendices A and B. For further details, the reader is encouraged to consult Chung [**9**] or D. Williams [**38**].

Regarding general notation used throughout this book, we note that if X is a real-valued random variable defined on a probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$, we shall sometimes use the notation $X \in \mathcal{G}$ to indicate that X is $\mathcal{G}$-measurable. Also, two random variables will be considered equal if they are equal almost surely, and two stochastic processes will be considered equal if they are indistinguishable (see Appendices B and C). In this chapter and the next, we restrict attention to finite sample spaces and consider only probability measures that give positive probability to each individual outcome. Consequently, in these chapters, equality of random variables and indistinguishability of discrete time stochastic processes actually entail equality surely.

2.1. Binomial or CRR Model

The CRR model is a simple discrete time model for a financial market. There are finitely many times $t = 0, 1, 2, \ldots, T$ (where $T < \infty$ is a positive

integer and successive times are successive integers). At each of these times the values of two assets can be observed. One asset is a risky security called a stock, and the other is a riskless security called a bond.

The *bond* is assumed to yield a constant rate of return $r \geq 0$ over each time period $(t-1, t]$; and so assuming the bond is valued at \$1 at time zero, the value of the bond at time t is given by

$$B_t = (1+r)^t, \quad t = 0, 1, \ldots, T. \tag{2.1}$$

We measure holdings in the bond in units, where the value of one unit at time t is B_t.

The *stock* price process is modeled as an exponential random walk such that S_0 is a strictly positive constant and

$$S_t = S_{t-1}\xi_t, \quad t = 1, 2, \ldots, T, \tag{2.2}$$

where $\{\xi_t, t = 1, 2, \ldots, T\}$ is a sequence of independent and identically distributed random variables with

$$P(\xi_t = u) = p = 1 - P(\xi_t = d), \tag{2.3}$$

where $p \in (0, 1)$ and $0 < d < 1+r < u$. The last two conditions are assumed to avoid arbitrage opportunities in the primary market model and to ensure stock prices are random and strictly positive. Note that

$$S_t = S_0 \prod_{i=1}^{t} \xi_i, \quad t = 0, 1, \ldots, T. \tag{2.4}$$

(Here we assume that an empty product is defined to take the value 1.) One may represent the possible paths that S_t follows using a binary tree (see Figure 1). Note that there are only three distinct values for S_2; i.e., the two middle dots in Figure 1 have the same value for S_2. The points have been drawn as two distinct points to emphasize the fact that they may be reached by different paths, that is, through different values for the sequence S_0, S_1, S_2. For pricing and hedging of some contingent claims under the binomial model (e.g., a European contingent claim whose payoff depends only on the terminal value of the stock price or an American contingent claim whose payoff at time t depends only on the stock price at that time), one may use a so-called recombining tree in which only the distinct values of S_i are indicated (in particular, the two middle dots associated with values of S_2 in Figure 1 are identified). However, in general one needs the full path structure of the stock price process to price and hedge contingent claims; cf. Exercise 3 at the end of this chapter.

For concreteness, and without loss of generality, we assume that the probability space $(\Omega, \mathcal{F}, P)$ on which our random variables are defined is such that Ω is the finite set of 2^T possible outcomes for the values of the

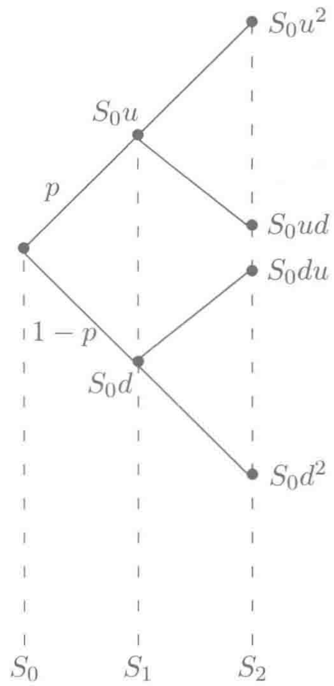

Figure 1. Binary tree for $T = 2$

stock price $(T+1)$-tuple, $(S_0, S_1, S_2, \ldots, S_T)$; $\mathcal{F}$ is the σ-algebra consisting of all possible subsets of Ω; and P is the probability measure on $(\Omega, \mathcal{F})$ associated with the Bernoulli probability p. Then, for example,

$$P((S_0, S_1, \ldots, S_T) = (S_0, S_0u, S_0u^2, \ldots, S_0u^T)) = p^T. \tag{2.5}$$

Indeed, each of the 2^T possible outcomes in Ω has strictly positive probability under P. The expectation operator under P will be denoted by $E[\cdot]$.

To describe the information available to the investor at time t, we introduce the σ-algebra generated by the stock prices up to and including time t; i.e., let

$$\mathcal{F}_t = \sigma\{S_0, S_1, \ldots, S_t\}, \quad t = 0, 1, \ldots, T. \tag{2.6}$$

In particular, with our concrete probability space, $\mathcal{F}_T = \mathcal{F}$. The family $\{\mathcal{F}_t, t = 0, 1, \ldots, T\}$ is called a *filtration*. We will often write it simply as $\{\mathcal{F}_t\}$.

A *trading strategy* (in the primary market) is a collection of pairs of random variables

$$\phi = \{(\alpha_t, \beta_t), t = 1, 2, \ldots, T\} \tag{2.7}$$

where the random variable α_t represents the number of shares of stock to be held over the time interval $(t-1, t]$ and the random variable β_t represents

the number of units of the bond to be held over the time interval $(t-1, t]$. For simplicity, we allow α_t, β_t to take any values in $\mathbb{R}$. In particular, we do not restrict to integer numbers of shares of stock or units of the bond, and we allow short selling of shares ($\alpha_t < 0$) and borrowing ($\beta_t < 0$). We think of trading occurring at time $t-1$ to determine the portfolio holdings (α_t, β_t) until the next trading time t. To avoid strategies that anticipate the future, it is assumed that α_t, β_t are $\mathcal{F}_{t-1}$-measurable random variables for $t = 1, 2, \ldots, T$. (In this discrete model setting, this simply means that α_t and β_t can be expressed as real-valued functions of $(S_0, S_1, \ldots, S_{t-1})$. In fact, the dependence on S_0 is trivial since S_0 is assumed to be a constant.) Thus, the holdings in stock and bond over the time period $(t-1, t]$ can depend only on the stock prices observed up to and including time $t-1$. We will restrict attention here to *self-financing* trading strategies, namely, those trading strategies ϕ such that

$$\alpha_t S_t + \beta_t B_t = \alpha_{t+1} S_t + \beta_{t+1} B_t, \quad t = 1, 2, \ldots, T-1, \tag{2.8}$$

and the investor's initial wealth is equal to

$$\alpha_1 S_0 + \beta_1 B_0. \tag{2.9}$$

We will simply refer to these as trading strategies rather than using the longer term "self-financing trading strategies". We say that a trading strategy ϕ represents a portfolio whose *value* at time t is given by $V_t(\phi)$, where

$$V_0(\phi) = \alpha_1 S_0 + \beta_1 B_0, \tag{2.10}$$

$$V_t(\phi) = \alpha_t S_t + \beta_t B_t, \quad t = 1, 2, \ldots, T. \tag{2.11}$$

An *arbitrage opportunity* (in the primary market) is a trading strategy ϕ such that $V_0(\phi) = 0$, $V_T(\phi) \geq 0$ and $E[V_T(\phi)] > 0$. We note that, in the presence of the preceding conditions, the last condition is equivalent to $P(V_T(\phi) > 0) > 0$.

2.2. Pricing a European Contingent Claim

A *European contingent claim* (ECC) is represented by an $\mathcal{F}_T$-measurable random variable X. This is interpreted to mean that the value of the contingent claim at the expiration time T is given by X. For example, a *European call option* with strike price $K \in (0, \infty)$ and expiration date T is represented by $X = (S_T - K)^+ \equiv \max\{0, S_T - K\}$. The meaning of X in this case is as follows. If $S_T > K$, the holder of the European call option will exercise it at T and, after selling a share of stock just purchased for K, will make a profit of $S_T - K$ (ignoring whatever was paid for the option in the first place). Thus, in this case, the value of the option at T is $S_T - K$. On the other hand, if $S_T \leq K$, the holder of the option will not exercise it at T and

its value is 0 at T. Similarly, a European put option with the same strike price and expiration date is represented by $X = (K - S_T)^+$.

A *replicating (or hedging) strategy* for a European contingent claim X is a trading strategy ϕ such that $V_T(\phi) = X$. If there exists such a replicating strategy, the contingent claim is said to be *attainable* (or redundant).

In this section, we derive the (initial) arbitrage free price for a European contingent claim. Existence of such a price depends on the existence of a so-called risk neutral probability, and uniqueness depends on there being a replicating strategy for the contingent claim.

2.2.1. Single Period Case. We begin by examining the single period case where $T = 1$. For this case, we first show that there is a replicating strategy for any European contingent claim X. Given X, we seek a trading strategy $\phi = (\alpha_1, \beta_1)$, where α_1 and β_1 are constants such that

$$V_1(\phi) \equiv \alpha_1 S_1 + \beta_1 B_1 = X. \tag{2.12}$$

Now S_1 has two possible values, $S_0 u, S_0 d$, and X is a function of S_1, since it is $\mathcal{F}_1 = \sigma\{S_0, S_1\}$-measurable. Let X^u denote the value of X when $S_1 = S_0 u$ and X^d denote the value of X when $S_1 = S_0 d$. Then considering these two possible outcomes, (2.12) yields two equations for the two deterministic unknowns α_1, β_1:

$$\alpha_1 S_0 u + \beta_1(1+r) = X^u \tag{2.13}$$
$$\alpha_1 S_0 d + \beta_1(1+r) = X^d. \tag{2.14}$$

Solving for α_1, β_1 yields

$$\alpha_1 = \frac{X^u - X^d}{(u-d)S_0}, \tag{2.15}$$
$$\beta_1 = \frac{1}{1+r}\left(\frac{uX^d - dX^u}{u-d}\right). \tag{2.16}$$

The right member of (2.15) is sometimes written informally as $\frac{\delta X}{\delta S}$. The initial wealth needed to finance this strategy (sometimes called the manufacturing cost of the contingent claim) is

$$\begin{aligned} V_0 &= \alpha_1 S_0 + \beta_1 B_0 \\ &= \frac{1}{(1+r)(u-d)}\left((1+r-d)X^u + (u-(1+r))X^d\right) \\ &= \frac{1}{1+r}\left(p^* X^u + (1-p^*)X^d\right) \\ &= E^{p^*}[X^*], \end{aligned} \tag{2.17}$$

where $X^* = X/(1+r)$, $p^* = \frac{1+r-d}{u-d}$, and $E^{p^*}[\cdot]$ denotes the expectation operator with p replaced by p^*. (For brevity, in this chapter, when probabilities or (conditional) expectations are those obtained by using p^* in place of p, we shall simply say that the probability or expectation is computed under p^*.)

Note that $p^* \in (0,1)$ and so, just as for p, each of the two possible outcomes for (S_0, S_1) has positive probability under p^*. Furthermore, $E^{p^*}[S_1] = (1+r)S_0$, and so the discounted stock price process

$$\left\{S_0, \frac{1}{1+r}S_1\right\}$$

is a martingale under p^* (relative to the filtration $\{\mathcal{F}_t\}$). Thus, under p^*, the average rate of return of the risky asset is the same as that of the riskless asset. For this reason p^* is called the *risk neutral probability.* (A person is said to be "risk averse" if the person prefers the expected value of a payoff to the random payoff itself. A person is said to be "risk preferring" if the person prefers the random payoff to the expected value of the payoff. A person is "risk neutral" if the person is neither risk averse nor risk preferring; in other words the person is indifferent, having no preference for the expected payoff versus the random payoff. The probability p^* is called a *risk neutral probability* because under p^*, a risk neutral investor would be indifferent to the choice at time zero of investing S_0 in stock or bond, since both investments have the same expected payoff at time 1 under p^*.)

It is important to realize that computing expectations under p^* is a mathematical device. We are not assuming that the stock price actually moves according to this probability. That is, p^* may be unrelated to the subjective probability p that we associate with the binomial model for movements in the stock price.

For the next theorem, we need the notion of an arbitrage opportunity in the market consisting of the stock, bond and contingent claim. For this, we suppose that the price of the contingent claim at time zero is C_0, a constant. A *trading strategy* in stock, bond and the contingent claim is a triple $\psi = (\alpha_1, \beta_1, \gamma_1)$ of $\mathcal{F}_0$-measurable random variables (these will actually be constants), where α_1 represents the number of shares of stock held over $(0,1]$, β_1 represents the number of units of the bond held over $(0,1]$, and γ_1 represents the number of units of the contingent claim held over $(0,1]$. The initial value of the portfolio associated with ψ is $V_0(\psi) = \alpha_1 S_0 + \beta_1 B_0 + \gamma_1 C_0$. The value of this portfolio at time one is $V_1(\psi) = \alpha_1 S_1 + \beta_1 B_1 + \gamma_1 X$. An *arbitrage opportunity* in the stock-bond-contingent claim market is a trading strategy $\psi = (\alpha_1, \beta_1, \gamma_1)$ such that $V_0(\psi) = 0$, $V_1(\psi) \geq 0$ and $E[V_1(\psi)] > 0$.

Theorem 2.2.1. *$V_0 = E^{p^*}[X^*]$ is the unique arbitrage free initial price for the European contingent claim X.*

Proof. Let $\phi^* = (\alpha_1^*, \beta_1^*)$ denote the replicating strategy (in stock and bond) for the contingent claim X; cf. (2.15)–(2.16).

First we show that if the initial price C_0 of the contingent claim is anything other than V_0, then there is an arbitrage opportunity in the stock-bond-contingent claim market. Suppose $C_0 > V_0$. Then an investor starting with zero initial wealth could sell one contingent claim ($\gamma_1 = -1$) for C_0, invest V_0 in the replicating strategy $\phi^* = (\alpha_1^*, \beta_1^*)$ and invest the remainder, $C_0 - V_0$, in bond. Thus, the trading strategy in stock, bond and contingent claim would be $(\alpha_1^*, \beta_1^* + C_0 - V_0, -1)$. This has an initial value of zero, and its value at time one is

$$\alpha_1^* S_1 + \beta_1^* B_1 + (C_0 - V_0)B_1 - X. \tag{2.18}$$

But the strategy (α_1^*, β_1^*) was chosen so that

$$\alpha_1^* S_1 + \beta_1^* B_1 = X, \tag{2.19}$$

and so it follows that the value at time one of the stock-bond-contingent claim portfolio is

$$(C_0 - V_0)B_1 > 0. \tag{2.20}$$

Thus, this represents an arbitrage opportunity. Similarly, if $C_0 < V_0$, then the investor can use the strategy $(-\alpha_1^*, -\beta_1^* + V_0 - C_0, 1)$ to create an arbitrage opportunity.

Now we show that if $C_0 = V_0$, then there is no arbitrage opportunity in the stock-bond-contingent claim market. Suppose that $\psi = (\alpha_1, \beta_1, \gamma_1)$ is a trading strategy in stock, bond and the contingent claim, with an initial value $V_0(\psi) = \alpha_1 S_0 + \beta_1 + \gamma_1 C_0$ of zero and non-negative value $V_1(\psi)$ at time one. The value of the portfolio at time one is

$$V_1(\psi) = \alpha_1 S_1 + \beta_1 B_1 + \gamma_1 X \tag{2.21}$$

and so

$$\begin{aligned} E^{p^*}[V_1(\psi)] &= \alpha_1 E^{p^*}[S_1] + \beta_1(1+r) + \gamma_1 E^{p^*}[X] \\ &= \alpha_1(1+r)S_0 + \beta_1(1+r) + \gamma_1(1+r)C_0 \\ &= (1+r)V_0(\psi) \\ &= 0, \end{aligned}$$

where we have used the fact that $\alpha_1, \beta_1, \gamma_1$ are constants (as they are $\mathcal{F}_0$-measurable), plus the martingale property of the discounted stock price process under p^* and the assumption that $C_0 = V_0 = E^{p^*}[X]/(1+r)$. Now, since $V_1(\psi) \geq 0$ and the probability associated with p^* gives positive probability to all possible outcomes, it follows from the equality above that $V_1(\psi) = 0$, and hence $E[V_1(\psi)] = 0$. Thus, there cannot be an arbitrage opportunity. $\square$

Examination of the proof of Theorem 2.2.1 reveals that existence of an arbitrage free price for the contingent claim depends on the martingale property of $\{S_0, \frac{1}{1+r}S_1\}$ under p^* and uniqueness of the price depends on the existence of a replicating strategy for the contingent claim. These aspects will hold true more generally.

2.2.2. Multi-Period Case. We now consider the general binomial model where T is any fixed positive integer. In this subsection p^* has the same value as in the single period case, namely, $p^* = \frac{1+r-d}{u-d}$. Just as in the single period case, under the probability associated with p or p^*, all of the 2^T possible outcomes for the $(T+1)$-tuple $(S_0, S_1, S_2, \ldots, S_T)$ have positive probability.

We first show that there is a replicating strategy for a European contingent claim X. For this, given X, we seek a trading strategy $\phi = \{(\alpha_t, \beta_t), t = 1, \ldots, T\}$ such that

$$V_T(\phi) \equiv \alpha_T S_T + \beta_T B_T = X. \tag{2.22}$$

This is developed by working backwards through the binary tree.

Let $V_T = X$. Since X is an $\mathcal{F}_T$-measurable random variable and the $(T+1)$-tuple $(S_0, S_1, \ldots, S_T)$ can take only finitely many possible values, it follows that

$$V_T = f(S_0, S_1, \ldots, S_T)$$

for some real-valued function f of $T+1$ variables. Firstly, suppose we condition on knowing the value of $S_0, S_1, \ldots, S_{T-1}$. Then the cost and associated trading strategy for manufacturing the contingent claim over the time period $(T-1, T]$ can be computed in a very similar manner to that for the single period model. Given $S_0, S_1, \ldots, S_{T-1}$, there are two possible values for V_T at time T, depending on whether $S_T = S_{T-1}u$ or $S_T = S_{T-1}d$. Denote these two values by V_T^u and V_T^d. In fact, $V_T^u = f(S_0, S_1, \ldots, S_{T-1}, S_{T-1}u)$ and $V_T^d = f(S_0, S_1, \ldots, S_{T-1}, S_{T-1}d)$. Note that these are $\mathcal{F}_{T-1}$-measurable random variables (the notation V_T^u, V_T^d hides this fact but has the advantage that it makes the formulas for the replicating strategy appear simpler). If the contingent claim is a European call option with strike price K and expiration date T, then $X = (S_T - K)^+$ and $V_T^u = (S_{T-1}u - K)^+$, $V_T^d = (S_{T-1}d - K)^+$.

Now, for any European contingent claim X, by similar analysis to that for the single period case, to ensure that $V_T(\phi) = X$ we obtain the following allocations for the time period $(T-1, T]$:

$$\alpha_T = \frac{V_T^u - V_T^d}{(u-d)S_{T-1}} \tag{2.23}$$

$$\beta_T = \frac{1}{(1+r)^T}\left(\frac{uV_T^d - dV_T^u}{u-d}\right) \tag{2.24}$$

and the capital required at time $T-1$ to finance these allocations in a self-financing manner is

$$
\begin{aligned}
V_{T-1} &= \frac{1}{1+r}\left(p^* V_T^u + (1-p^*)V_T^d\right) \\
&= \frac{1}{1+r}E^{p^*}[V_T \mid \mathcal{F}_{T-1}], \\
&= \frac{1}{1+r}E^{p^*}[X|\mathcal{F}_{T-1}],
\end{aligned} \tag{2.25}
$$

where $p^* = \frac{1+r-d}{u-d}$ and $E^{p^*}[\cdot \mid \mathcal{F}_{T-1}]$ denotes the conditional expectation, given $\mathcal{F}_{T-1}$, under p^*.

We can find a trading strategy $\phi = \{(\alpha_t, \beta_t), t = 1, 2, \ldots, T\}$ with associated value process $\{V_t(\phi), t = 0, 1, \ldots, T\}$ such that $V_T = X$ by proceeding inductively backwards through the binary tree as follows. For the induction step, fix $t \in \{1, \ldots, T-1\}$ and assume that (self-financing) allocations $(\alpha_{t+1}, \beta_{t+1}),\ldots,(\alpha_T, \beta_T)$ have been determined for the time periods $(t, t+1], \ldots, (T-1, T]$ with values $V_t,\ldots,V_{T-1}$, at times $t, t+1, \ldots, T-1$, respectively, such that $V_s = \frac{1}{(1+r)^{T-s}}E^{p^*}[X|\mathcal{F}_s]$ for $s = t, t+1, \ldots, T-1$, and the value of (α_T, β_T) at time T is $V_T = X$. Given $S_0, S_1, \ldots, S_{t-1}$, the holdings α_t and β_t for the time period $(t-1, t]$ are chosen so that the value associated with these holdings at time t is the same as that of the random variable V_t; i.e., letting V_t^u and V_t^d denote the two possible values of V_t given $S_0, S_1, \ldots, S_{t-1}$, define

$$\alpha_t = \frac{V_t^u - V_t^d}{(u-d)S_{t-1}}, \tag{2.26}$$

$$\beta_t = \frac{1}{(1+r)^t}\left(\frac{uV_t^d - dV_t^u}{u-d}\right). \tag{2.27}$$

One can readily check, using the induction hypothesis, that the capital needed at time $t-1$ to finance these holdings in a self-financing manner is

$$
\begin{aligned}
V_{t-1} &= \frac{1}{1+r}E^{p^*}[V_t \mid \mathcal{F}_{t-1}] \\
&= \frac{1}{(1+r)^{T-t+1}}E^{p^*}\left[E^{p^*}[X \mid \mathcal{F}_t] \mid \mathcal{F}_{t-1}\right] \\
&= \frac{1}{(1+r)^{T-t+1}}E^{p^*}[X \mid \mathcal{F}_{t-1}].
\end{aligned}
$$

Here we have used the tower property of conditional expectations; cf. Appendix A.

This completes the induction step and so it follows that the above procedure constructs a (self-financing) trading strategy, $\phi = \{(\alpha_t, \beta_t), t =$

$1, \ldots, T\}$ with value process $\{V_t(\phi), t = 0, 1, \ldots, T\}$ satisfying

$$V_t(\phi) = \frac{1}{(1+r)^{T-t}} E^{p^*}[X|\mathcal{F}_t] \tag{2.28}$$

for $t = 0, 1, \ldots, T$. In particular, $V_T(\phi) = X$ and

$$V_0 = \frac{1}{(1+r)^T} E^{p^*}[X]. \tag{2.29}$$

Here we have used the fact that $\mathcal{F}_0$ is trivial, being generated by the constant value S_0.

To prove that there exists an arbitrage free price for the contingent claim X, we need the following properties. For this, let $S_t^* = S_t/B_t = S_t/(1+r)^t$, $t = 0, 1, \ldots, T$. The process $S^* = \{S_t^*, t = 0, 1, \ldots, T\}$ is called the *discounted stock price process.*

Lemma 2.2.2. *The process $\{S_t^*, \mathcal{F}_t, t = 0, 1, \ldots, T\}$ is a martingale under p^*.*

Proof. Clearly S_t^* is $\mathcal{F}_t$-measurable, and S_t^* has finite mean for each t, since S_t^* takes only finitely many values. To verify the conditional expectation property, fix $t \in \{1, \ldots, T\}$. Then, using the fact that $\mathcal{F}_{t-1}$ is generated by $S_0, S_1, \ldots, S_{t-1}$ and ξ_t is independent of this σ-algebra, we have

$$\begin{aligned} E^{p^*}[S_t^* \mid \mathcal{F}_{t-1}] &= \frac{1}{(1+r)^t} E^{p^*}[S_{t-1}\xi_t \mid \mathcal{F}_{t-1}] \\ &= \frac{1}{(1+r)^t} S_{t-1} E^{p^*}[\xi_t] \\ &= \frac{S_{t-1}^*}{1+r} (p^* u + (1-p^*)d) \\ &= S_{t-1}^*, \end{aligned} \tag{2.30}$$

where we have used the definition of $p^* = \frac{1+r-d}{u-d}$ to obtain the last line. □

Lemma 2.2.3. *Let ϕ be a trading strategy in stock and bond with value process $\{V_t(\phi), t = 1, \ldots, T\}$. Consider the discounted value process $\{V_t^*(\phi) = V_t(\phi)/(1+r)^t, t = 0, 1, \ldots, T\}$. Then $\{V_t^*(\phi), \mathcal{F}_t, t = 0, 1, \ldots, T\}$ is a martingale under p^*.*

Proof. Let $\phi = \{(\alpha_t, \beta_t), t = 1, \ldots, T\}$. Note that $V_t^*(\phi) = \alpha_t S_t^* + \beta_t$ and $\alpha_t, \beta_t \in \mathcal{F}_{t-1}$ for $t = 1, \ldots, T$, and $V_0^*(\phi) = \alpha_1 S_0^* + \beta_1$. It is straightforward to show that $V_t^*(\phi)$ is $\mathcal{F}_t$-measurable and integrable for $t = 0, 1, \ldots, T$. Recall from Lemma 2.2.2 that $\{S_t^*, \mathcal{F}_t, t = 0, 1, \ldots, T\}$ is a martingale under p^*. Therefore, for $t = 1, \ldots, T$,

$$\begin{aligned} E^{p^*}[V_t^*(\phi) \mid \mathcal{F}_{t-1}] &= \alpha_t E^{p^*}[S_t^* \mid \mathcal{F}_{t-1}] + \beta_t \\ &= \alpha_t S_{t-1}^* + \beta_t. \end{aligned} \tag{2.31}$$

By factoring out $1/(1+r)^{t-1}$ and using the self-financing property of ϕ, it follows that for $t = 2, \ldots, T$

$$\begin{aligned} E^{P^*}\left[V_t^*(\phi) \mid \mathcal{F}_{t-1}\right] &= \frac{1}{(1+r)^{t-1}}\left(\alpha_t S_{t-1} + \beta_t B_{t-1}\right) \\ &= \frac{1}{(1+r)^{t-1}}\left(\alpha_{t-1} S_{t-1} + \beta_{t-1} B_{t-1}\right) \\ &= V_{t-1}^*(\phi). \end{aligned}$$

For $t = 1$,

$$\begin{aligned} E^{P^*}\left[V_1^*(\phi) \mid \mathcal{F}_0\right] &= \alpha_1 S_0 + \beta_1 B_0 \\ &= V_0(\phi) = V_0^*(\phi), \end{aligned}$$

by definition. Hence the desired martingale property holds. □

The following theorem is the multi-period analogue of the single period Theorem 2.2.1. Before proceeding to this, we must specify the notion of arbitrage in stock, bond and contingent claim to be used in the multi-period context. Let C_0 be the price charged for the contingent claim at time zero. Since we are specifying a price only for the contingent claim at time zero, trading in the contingent claim will be allowed only initially, whereas changes in the stock and bond holdings can occur at each of the times $t = 0, 1, \ldots, T-1$. A *trading strategy* in stock, bond and the contingent claim is a collection $\psi = \{(\alpha_t, \beta_t), t = 1, 2, \ldots, T; \gamma_1\}$ where for $t = 1, 2, \ldots, T$, α_t, β_t are $\mathcal{F}_{t-1}$-measurable random variables representing the holdings in stock and bond, respectively, to be held over the time interval $(t-1, t]$, and γ_1 is an $\mathcal{F}_0$-measurable random variable (actually a constant) representing the number of units of the contingent claim to be held over the time interval $(0, T]$. The trading strategy must be self-financing; i.e., its initial value is

$$V_0(\psi) = \alpha_1 S_0 + \beta_1 B_0 + \gamma_1 C_0, \tag{2.32}$$

and at each time $t = 1, \ldots, T-1$,

$$\alpha_t S_t + \beta_t B_t = \alpha_{t+1} S_t + \beta_{t+1} B_t. \tag{2.33}$$

The last equation does not involve the contingent claim since this is not traded after time zero. The value of the portfolio at time T is

$$V_T(\psi) = \alpha_T S_T + \beta_T B_T + \gamma_1 X. \tag{2.34}$$

An *arbitrage opportunity* in the stock-bond-contingent claim market is a trading strategy ψ such that $V_0(\psi) = 0$, $V_T(\psi) \geq 0$ and $E[V_T(\psi)] > 0$.

Theorem 2.2.4. *Let $X^* = X/(1+r)^T$. Then $V_0 = E^{P^*}[X^*]$ is the unique arbitrage free initial price for the European contingent claim X.*

Proof. The proof is very similar to that of Theorem 2.2.1.

Let $\phi^* = \{(\alpha_t^*, \beta_t^*), t = 1, \dots, T\}$ denote the replicating strategy for the contingent claim X obtained by applying the inductive procedure described above (cf. (2.26)–(2.27)). Then the value process $V = \{V_t(\phi^*), t = 0, 1, \dots, T\}$ for ϕ^* satisfies (2.28) for $t = 0, 1, 2, \dots, T$. Note that, by (2.29), V_0 is the initial value of this replicating strategy.

We use the existence of the replicating strategy to show that if the initial price C_0 of the contingent claim is not V_0, then there is an arbitrage opportunity in the stock-bond-contingent claim market. Suppose $C_0 > V_0$. Then an investor could sell one contingent claim initially, use V_0 of the proceeds to invest in the stock-bond replicating strategy ϕ^*, and buy $C_0 - V_0$ additional units of bond at time zero and hold them over the entire period $(0, T]$. Thus, the trading strategy is $\psi = \{(\alpha_t^*, \beta_t^* + C_0 - V_0), t = 1, 2, \dots, T; \gamma_1 = -1\}$. This has initial value $V_0(\psi) = \alpha_1^* S_0 + \beta_1^* + C_0 - V_0 - C_0 = 0$, since $B_0 = 1$ and $V_0 = V_0(\phi^*) = \alpha_1^* S_0 + \beta_1^*$. The value of this portfolio at time T is

$$\begin{aligned} V_T(\psi) &= \alpha_T^* S_T + \beta_T^* B_T + (C_0 - V_0)B_T - X \\ &= X + (C_0 - V_0)B_T - X = (C_0 - V_0)B_T > 0, \end{aligned} \tag{2.35}$$

where we have used the fact that $\{(\alpha_t^*, \beta_t^*), t = 1, 2, \dots, T\}$ is a replicating strategy for the contingent claim X and so has value X at time T. Thus, ψ is an arbitrage opportunity. Similarly, if $C_0 < V_0$, the strategy $-\psi$ is an arbitrage opportunity.

Now suppose $C_0 = V_0$. We show that there is no arbitrage opportunity in stock, bond and contingent claim trading. Let $\psi = \{(\alpha_t, \beta_t), t = 1, 2, \dots, T; \gamma_1\}$ be a trading strategy in stock, bond and contingent claim with an initial value of zero and final value $V_T(\psi)$ that is a non-negative random variable. Then,

$$0 = V_0(\psi) = \alpha_1 S_0 + \beta_1 B_0 + \gamma_1 C_0, \tag{2.36}$$

$$0 \le V_T(\psi) = \alpha_T S_T + \beta_T B_T + \gamma_1 X. \tag{2.37}$$

Note that $\phi = \{(\alpha_t, \beta_t), t = 1, 2, \dots, T\}$ is a trading strategy in stock and bond. Using the martingale property of $\{V_t^*(\phi), \mathcal{F}_t, t = 0, 1, \dots, T\}$ under p^*, on setting $V_t^*(\psi) = V_t(\psi)/(1+r)^t$ for $t = 0, 1, \dots, T$, we have

$$\begin{aligned} \frac{1}{(1+r)^T} E^{p^*}[V_T(\psi)] &= E^{p^*}[V_T^*(\psi)] \\ &= E^{p^*}[V_T^*(\phi)] + E^{p^*}[\gamma_1 X^*] \\ &= E^{p^*}[V_0^*(\phi)] + \gamma_1 E^{p^*}[X^*] \\ &= \alpha_1 S_0^* + \beta_1 B_0^* + \gamma_1 C_0 \\ &= V_0(\psi) = 0 \end{aligned}$$

where we have used the facts that $\gamma_1 \in \mathcal{F}_0$, $C_0 = E^{p^*}[X^*]$, $S_0^* = S_0$, $B_0^* = B_0$, and $V_0(\psi) = 0$. Since it was assumed that $V_T(\psi) \geq 0$ and the probability associated with p^* gives positive probability to all possible values of $V_T(\psi)$, it follows that $V_T(\psi) = 0$ and hence $E[V_T(\psi)] = 0$. Thus, there cannot be any arbitrage opportunity when $C_0 = V_0$. □

2.3. Pricing an American Contingent Claim

An *American contingent claim* (ACC) is represented by a (finite) sequence $Y = \{Y_t, t = 0, 1, \ldots, T\}$ of real-valued random variables such that Y_t is $\mathcal{F}_t$-measurable for $t = 0, 1, 2, \ldots, T$. The random variable Y_t, $t = 0, 1, 2, \ldots, T$, is interpreted as the payoff for the claim if the owner cashes it in at time t. The time at which the owner cashes in the claim is required to be a stopping time taking values in $\{0, 1, \ldots, T\}$; i.e., a random variable $\tau : \Omega \longrightarrow \{0, 1, \ldots, T\}$ such that $\{\tau = t\} \equiv \{\omega \in \Omega : \tau(\omega) = t\} \in \mathcal{F}_t$, $t = 0, 1, 2, \ldots, T$. For $s, t \in \{0, 1, \ldots, T\}$ such that $s \leq t$, let $\mathcal{T}_{[s,t]}$ denote the set of integer-valued stopping times that take values in the interval $[s, t]$. An example of an American contingent claim is an *American call option* with strike price K which has payoff $Y_t = (S_t - K)^+$ at time t, $t = 0, 1, 2, \ldots, T$. Note that if $S_t \leq K$, cashing in the contingent claim at time t has an equivalent payoff to that obtained by not exercising the option at all. We have adopted this convention so that we can use one framework for treating all contingent claims, including options and contracts.

An important feature of an American contingent claim is that the buyer and the seller of such a derivative have different actions available to them — the buyer may cash in the claim at any stopping time $\tau \in \mathcal{T}_{[0,T]}$, whereas the seller seeks protection from the risk associated with all possible choices of the stopping time τ by the buyer. As with the pricing of European contingent claims, for the pricing of American contingent claims, an essential role will be played by a trading strategy that hedges the risk for the seller of an American contingent claim. However, unlike the European contingent claim setting, the seller will not always be able to exactly replicate the payoff of the American contingent claim at all times t. Instead, the seller of an American contingent claim seeks a *superhedging strategy* which is a (self-financing) trading strategy ϕ whose value is at least as great as the payoff of the American contingent claim at each time t.

More precisely, let $Y = \{Y_t, t = 0, 1, \ldots, T\}$ be the payoff sequence for an American contingent claim (abbreviated as ACC). For $t = 0, 1, \ldots, T$, let U_t denote the minimum amount of wealth that the seller of the ACC must have at time t in order to ensure that the seller has enough to cover the payoff if the buyer cashes in the claim at some stopping time $\tau \in \mathcal{T}_{[t,T]}$. A superhedging strategy for the seller is a (self-financing) trading strategy

$\phi = \{(\alpha_t, \beta_t), t = 1, 2, \ldots, T\}$ with value $V_t(\phi)$ at time $t = 0, 1, \ldots, T$, such that $U_t \leq V_t(\phi)$ for $t = 0, 1, 2, \ldots T$. Such a ϕ can be constructed stepwise by proceeding backwards through the binary tree. In order to see this, note that $U_T = Y_T$. Conditioned on $\mathcal{F}_{T-1}$ (i.e., given $S_0, S_1, \ldots, S_{T-1}$), let U_T^u denote the amount that the seller of the ACC must cover at time T if the value of the stock at that time is $S_{T-1}u$ and let U_T^d denote the amount that the seller must cover at time T if the value of the stock at time T is $S_{T-1}d$. Thus, in the same manner as for the case of the European contingent claim, conditioned on $\mathcal{F}_{T-1}$, the minimum amount of wealth needed at time $T-1$ to cover the possible payoff of the ACC at time T is

$$\frac{1}{1+r} E^{p^*} \left[U_T \mid \mathcal{F}_{T-1} \right],$$

and the payoff at time T can be manufactured from this amount using $\mathcal{F}_{T-1}$-measurable allocations $(\tilde{\alpha}_T, \tilde{\beta}_T)$ to stock and bond over the time interval $(T-1, T]$ determined by the relations:

$$\tilde{\alpha}_T S_T + \tilde{\beta}_T B_T = U_T, \tag{2.38}$$

$$\tilde{\alpha}_T S_{T-1} + \tilde{\beta}_T B_{T-1} = \frac{1}{1+r} E^{p^*} \left[U_T \mid \mathcal{F}_{T-1} \right]. \tag{2.39}$$

Now, at time $T-1$, the seller must have at least

$$U_{T-1} = \max \left\{ Y_{T-1}, \frac{1}{1+r} E^{p^*} \left[U_T \mid \mathcal{F}_{T-1} \right] \right\} \tag{2.40}$$

in order to cover the payoff Y_{T-1} associated with the buyer possibly cashing in the claim at time $T-1$, and to have sufficient wealth to produce a value at time T that is at least as large as the time T-payoff $U_T = Y_T$ of the ACC. Let

$$\tilde{\delta}_T = U_{T-1} - \frac{1}{1+r} E^{p^*} \left[U_T \mid \mathcal{F}_{T-1} \right]. \tag{2.41}$$

Then $\tilde{\delta}_T$ is the excess wealth in U_{T-1} over what is needed to cover the claim payoff at time T. Note that there is no excess if $Y_{T-1} < U_{T-1}$. Proceeding backwards inductively through the tree and repeating a very similar argument at each stage to that applied for the time interval $(T-1, T]$, we see that for $t = 0, 1, \ldots, T-1$, conditioned on $\mathcal{F}_t$, the amount of wealth needed at time t to cover possible payoff of the claim in $[t, T]$ is

$$U_t = \max \left\{ Y_t, \frac{1}{1+r} E^{p^*} \left[U_{t+1} \mid \mathcal{F}_t \right] \right\}. \tag{2.42}$$

Moreover, conditioned on $\mathcal{F}_t$ (i.e., given $S_0, S_1, \ldots, S_t$) and letting U_{t+1}^u and U_{t+1}^d denote the two possible values of U_{t+1} corresponding to whether $S_{t+1} = S_t u$ or $S_{t+1} = S_t d$, the allocations in stock and bond, $(\tilde{\alpha}_{t+1}, \tilde{\beta}_{t+1})$ over

$(t, t+1]$, that have value at time t given by

$$\frac{1}{1+r} E^{p^*}\left[U_{t+1} \mid \mathcal{F}_t\right] \tag{2.43}$$

and value U_{t+1} at time $t+1$ are given by

$$\begin{aligned}
\tilde{\alpha}_{t+1} &= \frac{U_{t+1}^u - U_{t+1}^d}{(u-d)S_t} && (2.44)\\
\tilde{\beta}_{t+1} &= \frac{1}{(1+r)^{t+1}} \left(\frac{uU_{t+1}^d - dU_{t+1}^u}{u-d} \right). && (2.45)
\end{aligned}$$

Let

$$\tilde{\delta}_{t+1} = U_t - \frac{1}{1+r} E^{p^*}\left[U_{t+1} \mid \mathcal{F}_t\right]. \tag{2.46}$$

Then $\tilde{\delta}_{t+1}$ is the excess wealth in U_t over what is needed to cover possible payoff of the claim in $[t+1, T]$. Note from the definition of U_t that $\tilde{\delta}_{t+1} = 0$ if $Y_t < U_t$.

Assuming that $U_t, t = 0, 1, \ldots, T$ and $\tilde{\alpha}_t, \tilde{\beta}_t, \tilde{\delta}_t, t = 1, 2, \ldots, T$ are defined as above, let

$$\begin{aligned}
\alpha_t^* &= \tilde{\alpha}_t, && (2.47)\\
\beta_t^* &= \tilde{\beta}_t + \sum_{s=1}^{t} \frac{\tilde{\delta}_s}{B_{s-1}}, && (2.48)
\end{aligned}$$

for $t = 1, 2, \ldots, T$. For this, note that $\tilde{\delta}_s$ is the excess available at time $s-1$ over the amount $\frac{1}{1+r} E^{p^*}[U_s | \mathcal{F}_{s-1}]$ that is needed at time $s-1$ to fund $(\tilde{\alpha}_s, \tilde{\beta}_s)$. We think of this excess as being invested in the bond for the time period $(s-1, T]$. The cost of unit of bond is B_{s-1} at time $s-1$, and so the number of units of the bond bought with $\tilde{\delta}_s$ will be $\tilde{\delta}_s / B_{s-1}$.

By construction, for $t = 1, \ldots, T$,

$$\tilde{\alpha}_t S_t + \tilde{\beta}_t B_t = U_t.$$

The strategy $\phi^* = \{(\alpha_t^*, \beta_t^*), t = 1, \ldots, T\}$ for trading in stock and bond is self-financing since for $t = 1, \ldots, T-1$,

$$\begin{aligned}
\alpha_{t+1}^* S_t + \beta_{t+1}^* B_t &= \tilde{\alpha}_{t+1} S_t + \tilde{\beta}_{t+1} B_t + \left(\sum_{s=1}^{t+1} \frac{\tilde{\delta}_s}{B_{s-1}} \right) B_t \\
&= \frac{1}{1+r} E^{p^*} [U_{t+1} \mid \mathcal{F}_t] + \left(\sum_{s=1}^{t} \frac{\tilde{\delta}_s}{B_{s-1}} \right) B_t + \tilde{\delta}_{t+1} \\
&= U_t + \left(\sum_{s=1}^{t} \frac{\tilde{\delta}_s}{B_{s-1}} \right) B_t \\
&= \tilde{\alpha}_t S_t + \tilde{\beta}_t B_t + \left(\sum_{s=1}^{t} \frac{\tilde{\delta}_s}{B_{s-1}} \right) B_t \\
&= \alpha_t^* S_t + \beta_t^* B_t.
\end{aligned}$$

For the first equality above we used the definition of $(\alpha_{t+1}^*, \beta_{t+1}^*)$ (cf. (2.47)–(2.48)). For the second equality we used the fact that $(\tilde{\alpha}_{t+1}, \tilde{\beta}_{t+1})$ is designed to have the value given by (2.43) at time t, and we split off the last term from the sum in the previous line. For the third equality we used the definition (2.46) of $\tilde{\delta}_{t+1}$. For the fourth equality we used the fact that (α_t, β_t) is designed to have the value U_t at time t. The fifth equality simply uses the definition (2.47)–(2.48) of (α_t^*, β_t^*).

The initial value of the trading strategy ϕ^* is

$$\begin{aligned}
V_0(\phi^*) &= \alpha_1^* S_0 + \beta_1^* B_0 \\
&= \tilde{\alpha}_1 S_0 + \tilde{\beta}_1 B_0 + \tilde{\delta}_1 \\
&= \frac{1}{1+r} E^{p^*} [U_1] + U_0 - \frac{1}{1+r} E^{p^*} [U_1] \\
&= U_0.
\end{aligned}$$

Here we have used the fact that $\mathcal{F}_0$ is trivial, and so conditional expectations with respect to this σ-algebra simplify to ordinary expectations. The value of ϕ^* at time $t = 1, 2, \ldots, T$ is

$$V_t(\phi^*) = \tilde{\alpha}_t S_t + \tilde{\beta}_t B_t + \left(\sum_{s=1}^{t} \frac{\tilde{\delta}_s}{B_{s-1}} \right) B_t \geq U_t.$$

Consider the stopping time

$$\tau^* = \min\{s \geq 0 : U_s = Y_s\}. \tag{2.49}$$

Note that since $Y_s \leq U_s$ for $s = 0, 1, \ldots, T$, $Y_s < U_s$ for $0 \leq s < \tau^*$, and so $\tilde{\delta}_s = 0$ for $1 \leq s \leq \tau^*$, and hence

$$V_t(\phi^*) = U_t \quad \text{for } 0 \leq t \leq \tau^*. \tag{2.50}$$

For $t = 0, 1, \ldots, T$, define the discounted random variables

$$Y_t^* = (1+r)^{-t} Y_t, \qquad U_t^* = (1+r)^{-t} U_t. \tag{2.51}$$

Then for $t = 0, 1, \ldots, T-1$,

$$U_t^* = \max\left\{Y_t^*, E^{p^*}\left[U_{t+1}^* \mid \mathcal{F}_t\right]\right\}, \tag{2.52}$$

and $U_T^* = Y_T^*$. The process $\{U_t^*, t = 0, 1, \ldots, T\}$ is called the *Snell envelope* of $\{Y_t^*, t = 0, 1, \ldots, T\}$, and it has the following properties.

Lemma 2.3.1.

(i) *Under p^*, $U^* = \{U_t^*, \mathcal{F}_t, t = 0, 1, \ldots, T\}$ is the smallest supermartingale such that $U_t^* \geq Y_t^*$ for $t = 0, 1, \ldots, T$.*

(ii) *For $t = 0, 1, \ldots, T$,*

$$U_t^* = \max_{\tau \in \mathcal{T}_{[t,T]}} E^{p^*}\left[Y_\tau^* \mid \mathcal{F}_t\right]. \tag{2.53}$$

(iii) *For $t = 0, 1, \ldots, T$,*

$$\tau^*(t) \equiv \min\{v \geq t : U_v^* = Y_v^*\}$$

is an element of $\mathcal{T}_{[t,T]}$ that achieves the maximum in the right side of (2.53).

Proof. Throughout this proof, all expectations and conditional expectations are to be computed under p^*. For (i), the supermartingale property and the inequality are immediate consequences of (2.52). To show that U^* is the smallest supermartingale satisfying the inequality, suppose that

$$W = \{W_t, \mathcal{F}_t, t = 0, 1, \ldots, T\}$$

is another supermartingale such that $W_t \geq Y_t^*$, $t = 0, 1, \ldots, T$. Then $W_T \geq Y_T^* = U_T^*$. For a proof by backwards induction, suppose that $s \in \{0, 1, \ldots, T-1\}$ and $W_t \geq U_t^*$ for $t = s+1, \ldots, T$. Then by the supermartingale property of W and the induction assumption,

$$W_s \geq E^{p^*}\left[W_{s+1} \mid \mathcal{F}_s\right] \geq E^{p^*}\left[U_{s+1}^* \mid \mathcal{F}_s\right].$$

Since by assumption, $W_s \geq Y_s^*$, we have that W_s is greater than or equal to the maximum of Y_s^* and $E^{p^*}[U_{s+1}^* \mid \mathcal{F}_s]$, which equals U_s^*. This completes the induction step and so $W_t \geq U_t^*$ for $t = T, T-1, \ldots, 1, 0$.

Before proceeding with the main proof of parts (ii) and (iii), we first verify that $\tau^*(t)$ is a member of $\mathcal{T}_{[t,T]}$ for $t = 0, 1, \ldots, T$. Fix $t \in \{0, 1, \ldots, T\}$. Then $\tau^*(t) \geq t$ by the form of the definition and $\tau^*(t) \leq T$ since $U_T^* = Y_T^*$. Now Y_v^* and U_v^* are $\mathcal{F}_v$-measurable for each $v \in \{0, 1, \ldots, T\}$, and so it follows that for each $s \in \{t, t+1, \ldots, T\}$,

$$\{\tau^*(t) = s\} = \{Y_v^* < U_v^* \text{ for } v \in [t, s) \text{ and } Y_s^* = U_s^*\} \in \mathcal{F}_s.$$

Hence $\tau^*(t)$ is a stopping time.

We prove the remainder of (ii) and (iii) together, using backwards induction again. For $t = T$, both (ii) and (iii) are easy to show since $\mathcal{T}_{[T,T]} = \{T\}$, Y_T^* is $\mathcal{F}_T$-measurable, and $U_T^* = Y_T^*$. For the induction step, assume that for some $s \in \{0, 1, \ldots, T-1\}$ both (ii) and (iii) hold for $t = s+1, s+2, \ldots, T$. By (2.52) with $t = s$, and (ii) with $t = s+1$, for each $\sigma \in \mathcal{T}_{[s+1,T]}$ we have

$$\begin{aligned} U_s^* &\geq \max\left\{Y_s^*, E^{p^*}\left[E^{p^*}\left[Y_\sigma^* \mid \mathcal{F}_{s+1}\right] \mid \mathcal{F}_s\right]\right\} \\ &= \max\left\{Y_s^*, E^{p^*}\left[Y_\sigma^* \mid \mathcal{F}_s\right]\right\}. \end{aligned} \tag{2.54}$$

Here we have used the tower property of conditional expectations. For $\tau \in \mathcal{T}_{[s,T]}$,

$$Y_\tau^* = 1_{\{\tau=s\}} Y_s^* + 1_{\{\tau \geq s+1\}} Y_{\tau\vee(s+1)}^*,$$

where $\tau \vee (s+1) = \max\{\tau, s+1\}$. Since τ is a stopping time taking integer values in $[s, T]$, $1_{\{\tau\geq s+1\}} = 1_{\{\tau=s\}^c} \in \mathcal{F}_s$, where the superscript c denotes complement. Then using basic properties of conditional expectations we have

$$\begin{aligned} E^{p^*}\left[Y_\tau^* \mid \mathcal{F}_s\right] &= 1_{\{\tau=s\}} Y_s^* + 1_{\{\tau\geq s+1\}} E^{p^*}\left[Y_{\tau\vee(s+1)}^* \mid \mathcal{F}_s\right] \\ &\leq \max\left\{Y_s^*, E^{p^*}\left[Y_{\tau\vee(s+1)}^* \mid \mathcal{F}_s\right]\right\}. \end{aligned} \tag{2.55}$$

Since $\tau \vee (s+1) \in \mathcal{T}_{[s+1,T]}$, it follows from (2.54) and (2.55) that

$$U_s^* \geq E^{p^*}\left[Y_\tau^* \mid \mathcal{F}_s\right],$$

and since $\tau \in \mathcal{T}_{[s,T]}$ was arbitrary, we conclude that

$$U_s^* \geq \max_{\tau\in\mathcal{T}_{[s,T]}} E^{p^*}[Y_\tau^* \mid \mathcal{F}_s]. \tag{2.56}$$

The proof of (ii) and (iii) for $t = s$ will be complete once we verify that the maximum in the right member of (2.56) is achieved at $\tau = \tau^*(s)$ and the value of this maximum is U_s^*. By (2.52) with $t = s$ and (iii) with $t = s+1$, we have

$$\begin{aligned} U_s^* &= \max\left\{Y_s^*, E^{p^*}\left[E^{p^*}\left[Y_{\tau^*(s+1)}^* \mid \mathcal{F}_{s+1}\right] \mid \mathcal{F}_s\right]\right\} \\ &= \max\left\{Y_s^*, E^{p^*}\left[Y_{\tau^*(s+1)}^* \mid \mathcal{F}_s\right]\right\}. \end{aligned} \tag{2.57}$$

On $\{\tau^*(s) = s\}$, $Y_s^* = U_s^*$, and on $\{\tau^*(s) \geq s+1\}$, $Y_s^* < U_s^*$. Thus,

$$U_s^* = 1_{\{\tau^*(s)=s\}} Y_s^* + 1_{\{\tau^*(s)\geq s+1\}} E^{p^*}\left[Y_{\tau^*(s+1)}^* \mid \mathcal{F}_s\right].$$

Moreover, on $\{\tau^*(s) \geq s+1\}$, $\tau^*(s) = \tau^*(s+1)$ and so the above simplifies to

$$U_s^* = E^{p^*}\left[Y_{\tau^*(s)}^* \mid \mathcal{F}_s\right].$$

Here we have used the fact that $\tau^*(s)$ is a stopping time with integer values in $[s,T]$ and so $\{\tau^*(s) \geq s+1\} = \{\tau^*(s) = s\}^c \in \mathcal{F}_s$. This completes the proof of (ii) and (iii) for $t = s$, and so the desired result follows by induction. $\square$

Recall the definition of τ^* from (2.49) and note that

$$\tau^* = \tau^*(0).$$

Then we have the following.

Lemma 2.3.2. *Under p^*, $\{U^*_{t\wedge\tau^*}, \mathcal{F}_t, t = 0,1,\ldots,T\}$ is a martingale.*

Proof. Since τ^* is a stopping time, $U^*_{t\wedge\tau^*} \in \mathcal{F}_t$ for $t = 0,1,\ldots,T$ and the probability space is finite, the integrability of $U^*_{t\wedge\tau^*}$ is automatic for each t. Fix $t \in \{0,1,\ldots,T-1\}$. On $\{\tau^* > t\}$, $Y^*_t < U^*_t$ and thus $U^*_t = E^{p^*}\left[U^*_{t+1} \mid \mathcal{F}_t\right]$ there. Thus,

$$\begin{aligned} U^*_{(t+1)\wedge\tau^*} - U^*_{t\wedge\tau^*} &= 1_{\{\tau^*>t\}}\left(U^*_{t+1} - U^*_t\right) \\ &= 1_{\{\tau^*>t\}}\left(U^*_{t+1} - E^{p^*}\left[U^*_{t+1} \mid \mathcal{F}_t\right]\right), \end{aligned} \tag{2.58}$$

and then taking conditional expectations with respect to $\mathcal{F}_t$ under p^*, using the fact that $\{\tau^* > t\} \in \mathcal{F}_t$, yields

$$E^{p^*}[U^*_{(t+1)\wedge\tau^*} - U^*_{t\wedge\tau^*} \mid \mathcal{F}_t] = 0.$$

$\square$

We now argue that U_0 is the unique arbitrage free initial price for the American contingent claim. For this, we need the notion of an arbitrage in a market where the stock and bond can be traded and the American contingent claim (ACC) can be bought or sold at time zero. For such a market, let the initial price of the ACC be a constant C_0. There are two types of arbitrage opportunities: one for a seller and another for a buyer of the ACC. The *seller* of the ACC has an *arbitrage opportunity* if there is a trading strategy ϕ^s in stock and bond such that $V_0(\phi^s) = C_0$ and for all stopping times $\tau \in \mathcal{T}_{[0,T]}$,

$$V_\tau(\phi^s) - Y_\tau \geq 0 \qquad \text{and} \qquad E\left[V_\tau(\phi^s) - Y_\tau\right] > 0. \tag{2.59}$$

The *buyer* of the ACC has an *arbitrage opportunity* if there is a trading strategy ϕ^b in stock and bond such that $V_0(\phi^b) = -C_0$ and there exists a stopping time $\tau \in \mathcal{T}_{[0,T]}$ such that

$$V_\tau(\phi^b) + Y_\tau \geq 0 \qquad \text{and} \qquad E\left[V_\tau(\phi^b) + Y_\tau\right] > 0. \tag{2.60}$$

The price C_0 is arbitrage free if there is no arbitrage opportunity for a seller or buyer of the contingent claim at this price.

To take advantage of a seller's arbitrage opportunity, an investor could sell one ACC at time zero for C_0 and invest the proceeds C_0 according to the trading strategy ϕ^s until the claim is cashed in by the buyer at some stopping time τ. At time τ, the seller would give the buyer Y_τ to pay off the claim and have a resultant wealth at time τ of $V_\tau(\phi^s) - Y_\tau$. The seller could put this amount in bond for the time period $(\tau, T]$. This would result in a final value of $(V_\tau(\phi^s) - Y_\tau)(1+r)^{T-\tau}$ which is non-negative and is strictly positive with positive probability (under p or p^*).

To take advantage of a buyer's arbitrage opportunity, an investor could buy one ACC at time zero for C_0, and invest $-C_0$ according to ϕ^b until the time τ when the buyer cashes in the claim. The buyer would then have $V_\tau(\phi^b) + Y_\tau$ at time τ and could put this in bond for the time period $(\tau, T]$ so that the buyer's final wealth would be $(V_\tau(\phi^b) + Y_\tau)(1+r)^{T-\tau}$ which is non-negative and is strictly positive with positive probability.

Remark. Here, in interpreting the definition of an arbitrage opportunity for the seller or buyer of an American contingent claim, we have imagined that the seller or buyer would continue investing in the primary market once the ACC had been cashed in. An advantage of this interpretation is that it facilitates comparison of payoffs at a fixed time T. Alternatively, one could imagine that once the ACC is cashed in, the buyer and seller do not further invest in the primary market. This is equivalent to them investing in a bond with zero interest rate once the ACC has been cashed in. With either interpretation, the mathematical definition of an arbitrage opportunity is the same.

The following lemma will be used in showing that an initial price of U_0 for the ACC is arbitrage free.

Lemma 2.3.3. *Let ϕ be a trading strategy in stock and bond with discounted value process*

$$\{V_t^*(\phi) = V_t(\phi)/(1+r)^t, t = 0, 1, \ldots, T\}.$$

Then, for any $\tau \in \mathcal{T}_{[0,T]}$,

$$E^{p^*}\left[V_\tau^*(\phi)\right] = V_0^*(\phi). \tag{2.61}$$

Proof. By Lemma 2.2.3, $\{V_t^*(\phi), \mathcal{F}_t, t = 0, 1, \ldots, T\}$ is a martingale under p^*. Equation (2.61) then follows from Doob's stopping theorem since τ is a bounded stopping time. □

Theorem 2.3.4. *The unique arbitrage free initial price for the American contingent claim is U_0.*

Proof. We first show that the arbitrage free price cannot be anything other than U_0; i.e., we establish uniqueness of an arbitrage free initial price.

Suppose that $C_0 > U_0$. Then there is an arbitrage opportunity for the seller of the American contingent claim. To see this, let ϕ^s denote the trading strategy in stock and bond corresponding to investing U_0 according to the superhedging strategy ϕ^*, and, in addition, putting $C_0 - U_0$ in the bond at time zero and holding those units of bond until time T. Note that the initial value $V_0(\phi^s) = C_0$ and the value $V_t(\phi^*)$ of ϕ^* at time t is at least U_t for $t = 0, 1, \ldots, T$. Then for any stopping time $\tau \in \mathcal{T}_{[0,T]}$,

$$\begin{aligned} V_\tau(\phi^s) - Y_\tau &= V_\tau(\phi^*) + (C_0 - U_0)B_\tau - Y_\tau \\ &\geq U_\tau + (C_0 - U_0)B_\tau - Y_\tau \\ &\geq (C_0 - U_0)B_\tau, \end{aligned} \tag{2.62}$$

where $(C_0 - U_0)B_\tau > 0$. Thus, there is an arbitrage opportunity for a seller of the American contingent claim.

On the other hand, suppose that $C_0 < U_0$. Then there is an arbitrage opportunity for the buyer of the American contingent claim. To see this, let ϕ^b denote the trading strategy corresponding to investing $-U_0$ according to the negative $-\phi^*$ of the superhedging strategy ϕ^* and investing $U_0 - C_0$ in the bond for all time. Furthermore, consider the stopping time τ^* (viewed as the time at which the ACC should be cashed in). Note that $V_t(-\phi^*) = -V_t(\phi^*)$ for $t = 0, 1, \ldots, T$, and $V_0(\phi^b) = -C_0$. Then, by (2.50) and the definition of τ^*, we have that $V_{\tau^*}(\phi^*) = U_{\tau^*} = Y_{\tau^*}$, and so

$$V_{\tau^*}(\phi^b) + Y_{\tau^*} = -V_{\tau^*}(\phi^*) + (U_0 - C_0)B_{\tau^*} + Y_{\tau^*} = (U_0 - C_0)B_{\tau^*}.$$

Since $(U_0 - C_0)B_{\tau^*} > 0$, there is an arbitrage opportunity for a buyer of the American contingent claim.

Finally, suppose that $C_0 = U_0$. We need to show that $C_0 = U_0$ is arbitrage free; i.e., that there exists an arbitrage free initial price. We begin by showing that there is no arbitrage opportunity for a seller of an American contingent claim with $C_0 = U_0$. For a proof by contradiction, suppose that there exists a trading strategy ϕ^s such that $V_0(\phi^s) = U_0$ and for each $\tau \in \mathcal{T}_{[0,T]}$, (2.59) holds. Note that $\tau^* \in \mathcal{T}_{[0,T]}$. Then it follows from (2.59) with $\tau = \tau^*$ that $V_{\tau^*}(\phi^s) - Y_{\tau^*} \geq 0$ and that this inequality holds strictly with positive probability under p and hence under p^*. These same properties continue to hold after multiplying the random variable by the random discount factor $(1+r)^{-\tau^*}$, and so it follows that

$$E^{p^*}\left[V^*_{\tau^*}(\phi^s) - Y^*_{\tau^*}\right] > 0. \tag{2.63}$$

On the other hand, by (2.61), $E^{p^*}\left[V^*_{\tau^*}(\phi^s)\right] = V^*_0(\phi^s) = U^*_0$, and by (ii) and (iii) in Lemma 2.3.1, $E^{p^*}\left[Y^*_{\tau^*}\right] = U^*_0$. Combining these two properties, we obtain $E^{p^*}\left[V^*_{\tau^*}(\phi^s) - Y^*_{\tau^*}\right] = 0$, which contradicts (2.63). Therefore, no such ϕ^s exists and consequently there is no arbitrage opportunity for a seller of the American contingent claim.

Next we show that there is no arbitrage opportunity for a buyer of the American contingent claim with $C_0 = U_0$. For this, suppose that there exists a trading strategy ϕ^b such that $V_0(\phi^b) = -U_0$ and a stopping time $\tau \in \mathcal{T}_{[0,T]}$ such that (2.60) holds. For a contradiction, we will show that $E\left[V_\tau(\phi^b) + Y_\tau\right] \leq 0$, or equivalently that $E^{p^*}\left[V_\tau(\phi^b) + Y_\tau\right] \leq 0$, which in turn is equivalent to showing that $E^{p^*}\left[V_\tau^*(\phi^b) + Y_\tau^*\right] \leq 0$ (since we have assumed that $V_\tau(\phi^b) + Y_\tau \geq 0$). By (2.61), $E^{p^*}\left[V_\tau^*(\phi^b)\right] = V_0^*(\phi^b) = -U_0^*$. Moreover, by (ii) in Lemma 2.3.1, $E^{p^*}\left[Y_\tau^*\right] \leq U_0^*$. Therefore, $E^{p^*}\left[V_\tau^*(\phi^b) + Y_\tau^*\right] \leq 0$, and the desired contradiction is obtained. Consequently there is no arbitrage opportunity for a buyer of the American contingent claim. □

2.4. Exercises

1. Consider a single period CRR model with $S_0 = \$100$, $S_1 = \$200$ or $\$50$, $r = 0.25$.

(a) Find the arbitrage free initial price of a European call option for one share of stock where the strike price is $K = \$100$ and the exercise time $T = 1$.

(b) Find a hedging strategy that replicates the value of the option described in (a).

(c) Suppose the option in (a) is initially priced at \$1 above the arbitrage free price. Describe a strategy (for trading in stock, bond and the option) that is an arbitrage.

(d) What is the arbitrage free initial price for a put option with the same strike price and exercise time as the call option described in (a)?

2. Consider a CRR model with $T = 2$, $S_0 = \$100$, $S_1 = \$200$ or $S_1 = \$50$, and an associated European call option with strike price $K = \$80$ and exercise time $T = 2$. Assume that the risk free interest rate is $r = 0.1$.

(a) Draw the binary tree and compute the arbitrage free initial price of the European call option at time zero.

(b) Determine an explicit hedging strategy for this option.

(c) Suppose that the option in (a) is initially priced \$2 below the arbitrage free price. Describe a strategy (for trading in stock, bond and the option) that is an arbitrage.

(d) Try to automate the pricing of a European call option in a computer program where T, S_0, u, d and K are variables.

3. Consider the same CRR model as in Exercise 2 but with an associated European contingent claim whose value at its exercise time $T = 2$ is given by

$$X = \max\{S_0, S_1, S_2\}. \tag{2.64}$$

Answer questions (a)–(d) of Exercise 2 for this contingent claim.

4. Consider the multi-period CRR model described in the text and a European contingent claim X. Suppose that trading in the contingent claim is allowed at each of the times $t = 0, 1, \ldots, T-1$, where the price of the contingent claim at time t is given by an $\mathcal{F}_t$-measurable random variable C_t. Let ϕ^* be a replicating strategy (in stock and bond) for the contingent claim X. Show that

$$V_t(\phi^*) = \frac{1}{(1+r)^{T-t}} E^{p^*}[X \mid \mathcal{F}_t], \quad t = 0, 1, \ldots, T. \tag{2.65}$$

Formulate a notion of an arbitrage opportunity in the market consisting of stock, bond and contingent claim. Show that the process $C_t = V_t(\phi^*), t = 0, 1, \ldots, T-1$ defines the unique arbitrage free price process for the contingent claim.

5. Consider a CRR model with terminal time T. A European call option with strike price K and expiration date T has value $C_T = (S_T - K)^+$ at time T. A European put option with the same strike price and expiration date has value $P_T = (K - S_T)^+$ at time T. Note that $C_T - P_T = S_T - K$. Let $\{C_t, t = 0, 1, \ldots, T-1\}$ and $\{P_t, t = 0, 1, \ldots, T-1\}$ denote the arbitrage free price processes for the call and put options, respectively (cf. Exercise 4). Show that the following call-put parity relation holds:

$$C_t - P_t = S_t - K/(1+r)^{T-t}, \quad t = 0, 1, \ldots, T-1. \tag{2.66}$$

Note, you can show this using either a "no arbitrage" argument or the formulas for C_t, P_t resulting from Exercise 3.

6. Consider a CRR model with $T = 2$, $S_0 = \$100$, $S_1 = \$200$ or $S_1 = \$50$ (the same model as in Exercise 2). Now consider an American put option with strike price $K = \$120$. Assume that the risk free interest rate is $r = 0.1$.

(a) Use a binary tree to compute the arbitrage free initial price of the American put option.

(b) Determine an explicit superhedging strategy ϕ^* for this option.

(c) Suppose that you can buy the American put option at time zero for \$1 less than its arbitrage free price. Explicitly describe a strategy that yields an arbitrage opportunity for a buyer of the American put option.

(d) Try to automate the arbitrage free pricing of an American put option in a computer program where T, S_0, u, d and K are variables.

7. Prove that the arbitrage free initial price for an American call option is the same as the arbitrage free initial price for a European call option. Furthermore, show that with this initial (arbitrage free) price, if the buyer of an American call option waits until the expiration date to exercise the option, this prevents the seller from making a risk free profit (that is, it prevents a seller's arbitrage).

Finite Market Model

The binomial model considered in the previous chapter is an example of a finite market model. In that example, we saw that the existence of both a risk neutral probability and a replicating strategy played a key role in justifying the unique arbitrage free price for any European contingent claim. In this chapter, we extend that idea to the pricing of European contingent claims in a general finite market model. We characterize those finite market models in which there is a risk neutral probability and in which all European contingent claims can be replicated. Indeed, we will prove the first fundamental theorem of asset pricing, which shows the equivalence of the absence of arbitrage in a finite market model to the existence of a risk neutral probability. We will also prove the second fundamental theorem of asset pricing, which shows that all European contingent claims in a finite market model without arbitrage can be replicated if and only if there is a unique risk neutral probability. Then, assuming there is such a unique risk neutral probability and allowing trading of the European contingent claim at all times, we show that there is a unique arbitrage free price process for each European contingent claim. These results have their origins in the 1979 seminal paper of Harrison and Kreps [**20**]. In the binomial model, there is a unique risk neutral probability and hence there is a unique arbitrage free price process for every European contingent claim (cf. Exercise 4 of Chapter 2). We round out this chapter with a discussion of (single period) markets in which there is more than one risk neutral probability. Although such markets have no arbitrage opportunities, there are some European contingent claims that are not replicable; these are called *incomplete* markets.

3.1. Definition of the Finite Market Model

Throughout this chapter we will be working within the framework of the following discrete time, finite state market model. For short we will call this a finite market model.

Let $(\Omega, \mathcal{F}, P)$ be a probability space where Ω is a finite set of possible outcomes, $\mathcal{F}$ is the σ-algebra consisting of all subsets of Ω and P is a probability measure on $(\Omega, \mathcal{F})$ such that $P(\{\omega\}) > 0$ for all $\omega \in \Omega$. Expectations under P will be written simply as $E[\cdot]$. Whenever another probability is to be used, this will be explicitly indicated in the notation.

We assume that there are finitely many times $t = 0, 1, 2, \ldots, T$ (where $T < \infty$ is a positive integer and successive times are successive integers). At each of these times the values of $d + 1$ assets can be observed. Here d is a strictly positive integer. One asset is a riskless security called a bond, and the other d assets are risky securities called stocks.

A σ-algebra $\mathcal{F}_t \subset \mathcal{F}$ describes the information available to an investor at time t. It is assumed that $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_T$ and $\mathcal{F}_T = \mathcal{F}$. The collection $\{\mathcal{F}_t, t = 0, 1, \ldots, T\}$ is called a filtration.

The *bond* (asset labelled 0) is assumed to have price process $S^0 = \{S_t^0, t = 0, 1, \ldots, T\}$, where S_t^0 denotes the price of the bond at time t. We assume that for each t, $S_t^0 > 0$ and S_t^0 is deterministic (i.e., $S_t^0 \in \mathcal{F}_0$). For example, if the bond has an interest rate of $r \geq 0$ per unit of time, then $S_t^0 = (1 + r)^t$ for all t. The bond is considered to be a "numeraire"; i.e., it tells us what a dollar at time 0 is worth (due, amongst other things, to the effects of inflation) at time t.

The d *stocks* are assumed to have price processes $S^1, \ldots, S^d$, where S_t^i is the price of the i^{th} stock at time t. It is assumed that S_t^i is an $\mathcal{F}_t$-measurable random variable for $i = 1, \ldots, d$ and $t = 0, 1, \ldots, T$. Note that since Ω is finite, $S_t = (S_t^0, S_t^1, \ldots, S_t^d)$, $t = 0, 1, \ldots, T$, can take on at most finitely many values. It follows that in the development below, all of the expectations we write will be automatically finite.

A *trading strategy* (in the finite market model) is a collection of $(d+1)$-dimensional vectors indexed by $t = 1, \ldots, T$:

$$\phi = \{\phi_t, t = 1, \ldots, T\}, \tag{3.1}$$

where for each $t \in \{1, \ldots, T\}$, $\phi_t = (\phi_t^0, \phi_t^1, \ldots, \phi_t^d)$ is such that ϕ_t^i is a real-valued $\mathcal{F}_{t-1}$-measurable random variable for $i = 0, 1, \ldots, d$. We regard ϕ_t^i as representing the number of "shares" of asset i to be held over the time interval $(t-1, t]$. In particular, ϕ_t^0 denotes the number of units of the bond to be held over this interval, and ϕ_t^i denotes the number of shares of stock i to be held over the interval for $i = 1, \ldots, d$. A positive value for ϕ_t^i indicates

that one buys that number of shares of asset i, at a price of S_{t-1}^i per share, and holds them over the interval $(t-1, t]$. A negative value for ϕ_t^i indicates that asset i will be "sold short". For example, if $\phi_t^i = -1$, one is effectively borrowing the value S_{t-1}^i of asset i at time $t-1$ with the understanding that the cost to repay this loan at time t is the value of one share of asset i at time t; i.e., S_t^i. We will restrict attention to *self-financing* trading strategies, namely, those trading strategies ϕ such that the investor's initial wealth is given by

$$\phi_1 \cdot S_0, \tag{3.2}$$

and

$$\phi_t \cdot S_t = \phi_{t+1} \cdot S_t, \quad t = 1, \ldots, T-1, \tag{3.3}$$

where $\cdot$ denotes the dot product in $\mathbb{R}^{d+1}$. In this chapter, the term "trading strategy" will always mean self-financing trading strategy.

The *initial value* of a trading strategy ϕ is $V_0(\phi) = \phi_1 \cdot S_0$ and its *value* at time $t \in \{1, \ldots, T\}$ is

$$V_t(\phi) \equiv \phi_t \cdot S_t = \sum_{i=0}^{d} \phi_t^i S_t^i. \tag{3.4}$$

Using the self-financing property we also have that

$$V_t(\phi) = \phi_{t+1} \cdot S_t, \quad t = 0, 1, \ldots, T-1. \tag{3.5}$$

The *gains process* associated with a trading strategy ϕ is defined by

$$G_t(\phi) = V_t(\phi) - V_0(\phi), \quad t = 0, 1, \ldots, T. \tag{3.6}$$

Using the equivalent forms for $V_s(\phi)$ that come from the self-financing property of ϕ, we can rewrite this process as follows for $t = 1, \ldots, T$:

$$\begin{aligned} G_t(\phi) &= \sum_{s=1}^{t} (V_s(\phi) - V_{s-1}(\phi)) \\ &= \sum_{s=1}^{t} (\phi_s \cdot S_s - \phi_s \cdot S_{s-1}) \\ &= \sum_{s=1}^{t} \phi_s \cdot (S_s - S_{s-1}) \\ &= \sum_{s=1}^{t} \phi_s \cdot \Delta S_s, \end{aligned} \tag{3.7}$$

where $\Delta S_s \equiv S_s - S_{s-1}$. In fact, the last expression is a discrete time stochastic integral (recall that ϕ_s is $\mathcal{F}_{s-1}$-measurable).

An *arbitrage opportunity* (in the finite market model) is a trading strategy ϕ such that

$$V_0(\phi) = 0, \qquad V_T(\phi) \geq 0, \qquad E[V_T(\phi)] > 0.$$

The finite market model is said to be *viable* if it has no arbitrage opportunities.

It will simplify computations to use *discounted asset price processes,* obtained by normalizing so that the value of a dollar at any time t is the same as it is at time 0. This is often called a change of numeraire. For $i = 0, 1, \ldots, d$, we define

$$S_t^{*,i} = \frac{S_t^i}{S_t^0}, \quad \text{for } t = 0, 1, \ldots, T.$$

Note that $S_t^{*,0} \equiv 1$ for all t. Then $S_t^* = (S_t^{*,0}, S_t^{*,1}, \ldots, S_t^{*,d})$ is the value of the vector of discounted asset prices at time t. We will refer to $S^* = \{S_t^*, t = 0, 1, \ldots, T\}$ as the (vector) discounted asset price process. The associated *discounted value process* for a trading strategy ϕ is defined by

$$V_t^*(\phi) \equiv \frac{V_t(\phi)}{S_t^0}, \quad t = 0, 1, \ldots, T, \tag{3.8}$$

and using (3.4) and (3.5) we see that

$$V_t^*(\phi) = \phi_t \cdot S_t^*, \quad t = 1, \ldots, T, \tag{3.9}$$

and

$$V_t^*(\phi) = \phi_{t+1} \cdot S_t^*, \quad t = 0, 1, \ldots, T-1. \tag{3.10}$$

The discounted gains process for ϕ is defined by

$$G_t^*(\phi) = V_t^*(\phi) - V_0^*(\phi), \quad t = 0, 1, \ldots, T, \tag{3.11}$$

and by very similar manipulations to those used in deriving (3.7), this can be reexpressed as $G_0^* = 0$ and

$$G_t^*(\phi) = \sum_{s=1}^{t} \phi_s \cdot \Delta S_s^*, \quad t = 1, \ldots, T, \tag{3.12}$$

where $\Delta S_s^* = S_s^* - S_{s-1}^*$. An advantage of this last expression is that it involves only the risky assets, since $\Delta S_s^{*,0} = 0$ for $s = 1, \ldots, T$.

3.2. First Fundamental Theorem of Asset Pricing

The following definitions will be needed to state the first fundamental theorem of asset pricing, which characterizes viable finite market models.

Definition 3.2.1. *Two probability measures Q and Q' on $(\Omega, \mathcal{F})$ are equivalent (or mutually absolutely continuous) provided for each $A \in \mathcal{F}$,*

$$Q(A) = 0 \quad \text{if and only if} \quad Q'(A) = 0. \tag{3.13}$$

Remark. In the finite market model, P gives positive probability to every $\omega \in \Omega$, and so for a probability measure P^* on $(\Omega, \mathcal{F})$, P is equivalent to P^* if and only if $P^*(\{\omega\}) > 0$ for all $\omega \in \Omega$.

Definition 3.2.2. *An equivalent martingale measure (abbreviated as EMM) is a probability measure P^* defined on $(\Omega, \mathcal{F})$ such that P^* is equivalent to P and S^* is a martingale under P^* (relative to the filtration $\{\mathcal{F}_t, t = 0, 1, \ldots, T\}$); i.e., for each $t \in \{1, \ldots, T\}$,*

$$E^{P^*}[S_t^* \,|\, \mathcal{F}_{t-1}] = S_{t-1}^*, \tag{3.14}$$

where $E^{P^*}[\cdot]$ denotes expectation under P^*, and the above equality is to be interpreted componentwise.

We note for future use that (3.14) is equivalent to

$$E^{P^*}[\Delta S_t^* \,|\, \mathcal{F}_{t-1}] = 0. \tag{3.15}$$

Remark. An equivalent martingale measure is sometimes also called a *risk neutral probability.* We will use the terms interchangeably.

Lemma 3.2.3. *Suppose that P^* is an equivalent martingale measure and ϕ is a trading strategy in the finite market model. Then $\{V_t^*(\phi), \mathcal{F}_t, t = 0, 1, \ldots, T\}$ is a martingale under P^*.*

Proof. For $t = 1, \ldots, T$, $V_t^*(\phi) = \phi_t \cdot S_t^* \in \mathcal{F}_t$ and for $t = 0$, $V_0^*(\phi) = \phi_1 \cdot S_0^* \in \mathcal{F}_0$, since $\phi_t \in \mathcal{F}_{t-1}$ for $t = 1, \ldots, T-1$ and $S_t^* \in \mathcal{F}_t$ for all t. For each t, $V_t^*(\phi)$ is integrable because Ω is a finite set. Finally, for each $t \in \{1, 2, \ldots, T\}$,

$$\begin{aligned} E^{P^*}\left[V_t^*(\phi) \,|\, \mathcal{F}_{t-1}\right] &= \phi_t \cdot E^{P^*}\left[S_t^* \,|\, \mathcal{F}_{t-1}\right] \\ &= \phi_t \cdot S_{t-1}^* = V_{t-1}^*(\phi), \end{aligned}$$

where we have used the fact that $\phi_t \in \mathcal{F}_{t-1}$, the martingale property (3.14) of S^* and the self-financing property (3.5) of ϕ. □

Theorem 3.2.4. *(First Fundamental Theorem of Asset Pricing) The finite market model is viable if and only if there exists an equivalent martingale measure P^*.*

Proof. We first prove the "if" part of the theorem. Suppose there exists an equivalent martingale measure P^*. For a proof by contradiction, suppose that ϕ is an arbitrage opportunity; that is, ϕ is a trading strategy with initial

value $V_0(\phi) = 0$, final value $V_T(\phi) \geq 0$, and $E[V_T(\phi)] > 0$. It follows that the discounted values satisfy $V_0^*(\phi) = 0$, $V_T^*(\phi) \geq 0$, and since P^* is equivalent to P, $E^{P^*}[V_T^*(\phi)] > 0$. By Lemma 3.2.3, $\{V_t^*(\phi), \mathcal{F}_t, t = 0, 1, \ldots, T\}$ is a martingale under P^* and so

$$E^{P^*}[V_T^*(\phi)] \quad = \quad E^{P^*}[V_0^*(\phi)].$$

However, the left member above is strictly positive whereas the right member is zero, which yields the desired contradiction. Thus, there cannot be an arbitrage opportunity in the finite market model, and hence the model is viable.

We now turn to proving the "only if" part of the theorem. For this, suppose that the finite market model is viable. Since Ω is a finite set, for any random variable Y defined on $(\Omega, \mathcal{F})$, by enumerating Ω as $\{\omega_1, \ldots, \omega_n\}$, we may view Y as $(Y(\omega_1), \ldots, Y(\omega_n)) \in \mathbb{R}^n$. Since $\mathcal{F}$ consists of all subsets of Ω, any point in $\mathbb{R}^n$ can be thought of as representing a real-valued random variable on $(\Omega, \mathcal{F})$. Thus, there is a one-to-one correspondence between points in $\mathbb{R}^n$ and (real-valued) random variables defined on Ω. Adopting this point of view for the terminal discounted gain random variables $G_T^*(\phi)$, we define

$$L \quad = \quad \{G_T^*(\phi) : \phi \text{ is a trading strategy such that } V_0(\phi) = 0\}.$$

Note that L is a linear space, since $G_T^*(\phi)$ is linear in ϕ and any linear combination of trading strategies with initial values of zero is again a trading strategy with the same initial value. Also, L is non-empty because the origin is contained in L. Let

$$D = \{Y \in \mathbb{R}^n : Y_i \geq 0 \text{ for } i = 1, \ldots, n \text{ and } Y_j > 0 \text{ for some } j\}.$$

Thus, D is the positive orthant in $\mathbb{R}^n$ with the origin removed.

Since the market is assumed to be viable, $L \cap D = \emptyset$. To see this, observe that if the latter were not true, there would be a trading strategy ϕ with $V_0(\phi) = 0$, $V_T^*(\phi) = G_T^*(\phi) \geq 0$ and $V_T^*(\phi)(\omega_i) > 0$ for at least one i, which would represent an arbitrage opportunity. Let

$$F = \left\{ Y \in D : \sum_{i=1}^{n} Y_i = 1 \right\}. \tag{3.16}$$

Then F is a convex, compact, non-empty subset of $\mathbb{R}^n$ and $L \cap F = \emptyset$ because $L \cap D = \emptyset$.

By applying the Separating Hyperplane Theorem 3.6.1 (see Section 3.6 for a statement and proof of this theorem), we see that there is a vector $Z \in \mathbb{R}^n \setminus \{0\}$ such that the hyperplane $H = \{Y \in \mathbb{R}^n : Y \cdot Z = 0\}$ contains L and $Z \cdot Y > 0$ for all $Y \in F$. By setting $Y_i = 1$ if $i = j$ and $Y_i = 0$ if $i \neq j$,

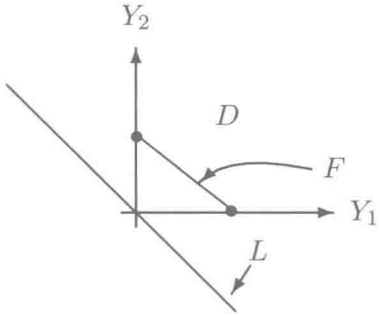

Figure 1. An example of the situation for $n = 2$

we see that $Z_j > 0$ for each $j \in \{1, \ldots, n\}$. Define

$$P^*(\{\omega_i\}) = \frac{Z_i}{\sum_{j=1}^n Z_j}, \quad i = 1, \ldots, n. \tag{3.17}$$

Then P^* is a probability measure on $(\Omega, \mathcal{F})$ and it is equivalent to P. Moreover, for any trading strategy ϕ such that $V_0(\phi) = 0$, we have

$$\begin{aligned} E^{P^*}[G_T^*(\phi)] &= \sum_{i=1}^n G_T^*(\phi)(\omega_i) \frac{Z_i}{\sum_{j=1}^n Z_j} \\ &= \frac{G_T^*(\phi) \cdot Z}{\sum_{j=1}^n Z_j} \\ &= 0, \end{aligned} \tag{3.18}$$

where the last line follows from the fact that Z is perpendicular to H, which contains L.

Note that $G_T^*(\phi)$ involves only $(\phi^1, \ldots, \phi^d)$. From Lemma 3.2.5, proved below, given $\hat{\phi}^1, \ldots, \hat{\phi}^d$, where for $i = 1, \ldots, d$, $\hat{\phi}^i = \{\hat{\phi}_t^i, t = 1, \ldots, T\}$ and $\hat{\phi}_t^i$ is a real-valued, $\mathcal{F}_{t-1}$-measurable random variable for each t, there is a unique time-ordered set of T real-valued random variables $\phi^0 = \{\phi_t^0, t = 1, \ldots, T\}$ such that $\phi \equiv \{(\phi_t^0, \hat{\phi}_t^1, \ldots, \hat{\phi}_t^d), t = 1, \ldots, T\}$ is a trading strategy with an initial value of zero. Upon substituting this in (3.18) and writing out the expression (cf. (3.12)) for $G_T^*(\phi)$, we see that

$$\begin{aligned} 0 = E^{P^*}[G_T^*(\phi)] &= E^{P^*}\left[\sum_{t=1}^T \phi_t \cdot \Delta S_t^*\right] \\ &= E^{P^*}\left[\sum_{t=1}^T \sum_{i=1}^d \hat{\phi}_t^i \Delta S_t^{*,i}\right]. \end{aligned} \tag{3.19}$$

For each fixed $i \in \{1, \ldots, d\}$, if we set $\hat{\phi}_t^j = 0$ for all t and $j \neq i$, we obtain

$$0 = E^{P^*}\left[\sum_{t=1}^{T} \phi_t^i \Delta S_t^{*,i}\right], \tag{3.20}$$

for each $\hat{\phi}^i = \{\hat{\phi}_t^i, t = 1, \ldots, T\}$ such that $\hat{\phi}_t^i$ is a real-valued $\mathcal{F}_{t-1}$-measurable random variable for each t. It then follows from Lemma 3.2.6, proved below, that for $i = 1, \ldots, d$, $S^{*,i}$ is a martingale under P^*. Hence, P^* is an equivalent martingale measure. □

The next two lemmas were used in the above proof of the first fundamental theorem of asset pricing. The first lemma shows that given (non-anticipating) holdings in the risky assets and an initial wealth, there is a unique sequence of holdings in the riskless asset that makes the associated trading strategy self-financing.

Lemma 3.2.5. *For $i = 1, \ldots, d$, let $\hat{\phi}^i = \{\hat{\phi}_t^i, t = 1, \ldots, T\}$ where $\hat{\phi}_t^i$ is a real-valued, $\mathcal{F}_{t-1}$-measurable random variable for $t = 1, \ldots, T$. For each real-valued $\mathcal{F}_0$-measurable random variable V_0, there exists a unique time-ordered set of T real-valued random variables $\phi^0 = \{\phi_t^0, t = 1, \ldots, T\}$ such that $\phi \equiv \{(\phi_t^0, \hat{\phi}_t^1, \ldots, \hat{\phi}_t^d), t = 1, \ldots, T\}$ is a trading strategy with an initial value of V_0.*

Proof. Fix $V_0 \in \mathcal{F}_0$. For ϕ to be self-financing at time zero, we must have (cf. (3.2)):

$$\phi_1 \cdot S_0 = V_0 \tag{3.21}$$

and since $\hat{\phi}_1^1, \ldots, \hat{\phi}_1^d$ are given, this will be satisfied if and only if

$$\phi_1^0 = (S_0^0)^{-1}\left(V_0 - \sum_{i=1}^{d} \hat{\phi}_1^i S_0^i\right). \tag{3.22}$$

Note that this $\phi_1^0 \in \mathcal{F}_0$. Thus, ϕ_1^0 is uniquely determined. For an induction, suppose that for some $1 \leq s \leq T-1$, ϕ_t^0, $t = 1, \ldots, s$, have been determined uniquely such that $\phi_t^0 \in \mathcal{F}_{t-1}$ for each $t = 1, \ldots, s$, (3.21) holds, and

$$\phi_t \cdot S_t = \phi_{t+1} \cdot S_t, \quad t = 1, \ldots, s-1. \tag{3.23}$$

Then, the self-financing property (3.23) holds for $t = s$ if and only if we have

$$\phi_{s+1}^0 = (S_s^0)^{-1}\left(\phi_s \cdot S_s - \sum_{i=1}^{d} \hat{\phi}_{s+1}^i S_s^i\right). \tag{3.24}$$

Note that this expression for ϕ_{s+1}^0 is $\mathcal{F}_s$-measurable. This establishes the induction step, and it follows that there is a unique ϕ^0 that makes ϕ a trading strategy with initial value V_0. □

Lemma 3.2.6. *Let $M = \{M_t, t = 0, 1, \ldots, T\}$ be a real-valued process such that $M_t \in \mathcal{F}_t$ for each t. Then, M is a martingale (relative to the filtration $\{\mathcal{F}_t\}$) if and only if*

$$E\left[\sum_{t=1}^{T} \eta_t \Delta M_t\right] = 0 \tag{3.25}$$

for all $\eta = \{\eta_t, t = 1, \ldots, T\}$ such that η_t is a real-valued $\mathcal{F}_{t-1}$-measurable random variable for $t = 1, \ldots, T$. Here, $\Delta M_t = M_t - M_{t-1}$ for $t = 1, \ldots, T$.

Remark. The sum $\sum_{t=1}^{T} \eta_t \Delta M_t$ is actually a discrete stochastic integral. If one extends η to a continuous time process by making it constant on $(t-1, t]$ with a value equal to that of η_t there, and one extends M to be constant on $[t-1, t)$ with a value equal to that of M_{t-1} there, then the sum is the same as the stochastic integral $\int_{[0,T]} \eta_t dM_t$.

Proof. Suppose M is a martingale. Let $\eta = \{\eta_t, t = 1, \ldots, T\}$ where η_t is a real-valued $\mathcal{F}_{t-1}$-measurable random variable for each t. Then, since $\eta_t \in \mathcal{F}_{t-1}$, we have

$$E\left[\sum_{t=1}^{T} \eta_t \Delta M_t\right] = \sum_{t=1}^{T} E\left[\eta_t E\left[\Delta M_t \mid \mathcal{F}_{t-1}\right]\right]. \tag{3.26}$$

Now, because M is a martingale we have $E\left[\Delta M_t \mid \mathcal{F}_{t-1}\right] = 0$, for $t = 1, \ldots, T$, and it follows that (3.25) holds.

Conversely, suppose that (3.25) holds for all $\eta = \{\eta_t, t = 1, \ldots, T\}$ where η_t is a real-valued $\mathcal{F}_{t-1}$-measurable random variable for each t. For fixed $s \in \{1, \ldots, T\}$ and $A \in \mathcal{F}_{s-1}$, let

$$\eta_t = \begin{cases} 0 & \text{for } t \neq s, \\ 1_A & \text{for } t = s. \end{cases}$$

Then $\eta_t \in \mathcal{F}_{t-1}$ for each t. Upon substituting this into (3.25), we obtain

$$E\left[1_A \Delta M_s\right] = 0. \tag{3.27}$$

Since $A \in \mathcal{F}_{s-1}$ was arbitrary, it follows that

$$E\left[M_s \mid \mathcal{F}_{s-1}\right] = M_{s-1}, \tag{3.28}$$

and then since s was arbitrary, it follows that M is a martingale. □

3.3. Second Fundamental Theorem of Asset Pricing

A *European contingent claim* is represented by an $\mathcal{F}_T$-measurable random variable X. This is interpreted to mean that the value (or payoff) of the contingent claim at the expiration time T is given by X. For example, a

European call option with strike price $K \in (0, \infty)$ and expiration date T that is based on the risky asset with price process S^1 is represented by $X = (S_T^1 - K)^+$. On the other hand, a look-back option usually depends on the recent history of a risky asset (see the Exercises for an example of such).

We will frequently refer to the random variable X, which represents a European contingent claim, as the European contingent claim (or ECC for short). For a European contingent claim X, we let $X^* = X/S_T^0$, the discounted value of X.

A *replicating (or hedging) strategy* for a European contingent claim X is a trading strategy ϕ such that $V_T(\phi) = X$. If there exists such a replicating strategy, the European contingent claim is said to be *attainable.*

The finite market model is said to be *complete* if all European contingent claims are attainable.

Theorem 3.3.1. *Suppose that the finite market model is viable and X is a replicable European contingent claim. Then the value process $\{V_t(\phi), t = 0, 1, \ldots, T\}$ is the same for all replicating strategies ϕ for X. Indeed, for any replicating strategy ϕ and any equivalent martingale measure P^*, we have*

$$V_t^*(\phi) = E^{P^*}[X^* \mid \mathcal{F}_t], \quad t = 0, 1, \ldots, T. \tag{3.29}$$

The right member of (3.29) defines the same stochastic process for all such P^*.

Remark. For any equivalent martingale measure P^*, P and P^* have the same sets of probability zero. Now, the only set of zero probability under P is the empty set, and so it follows that statements in this chapter that normally would hold only P-a.s., or P^*-a.s., in fact hold surely. In particular, two stochastic processes are indistinguishable (see Appendix B) under P if and only if they are indistinguishable under P^* if and only if they are equal surely. This accounts for the fact that in the statement of the above theorem, we did not include any a.s qualifiers.

Proof. By the first fundamental theorem of asset pricing, there is at least one equivalent martingale measure. Let P^* be such a measure. Let ϕ be a replicating strategy for X. Then, by the martingale property of $V^*(\phi)$ under P^* (see Lemma 3.2.3) and since $V_T^*(\phi) = X^*$, we have for $t = 0, 1, \ldots, T$,

$$\begin{aligned} V_t^*(\phi) &= E^{P^*}\left[V_T^*(\phi) \mid \mathcal{F}_t\right] \\ &= E^{P^*}\left[X^* \mid \mathcal{F}_t\right]. \end{aligned} \tag{3.30}$$

Since the last expression above does not depend upon ϕ, it follows that $\{V_t^*(\phi), t = 0, 1, \ldots, T\}$ and hence $\{V_t(\phi), t = 0, 1, \ldots, T\}$ does not vary with the particular choice of replicating strategy ϕ. Furthermore, since the

left member of (3.30) does not depend upon the particular choice of an EMM P^*, it follows that a stochastic process defined by the right member of (3.30) does not depend on the particular choice of P^*. □

Theorem 3.3.2. *(Second Fundamental Theorem of Asset Pricing) A viable finite market model is complete if and only if it admits a unique equivalent martingale measure.*

Proof. Suppose the market is viable and complete. By the first fundamental theorem of asset pricing, there is an equivalent martingale measure Q. Suppose there is another equivalent martingale measure $\tilde{Q}$. Fix $A \in \mathcal{F}_T$ and let $X = 1_A$. Suppose that ϕ is a replicating strategy for X. Then, by Theorem 3.3.1,

$$E^Q[X^*] = E^{\tilde{Q}}[X^*]. \tag{3.31}$$

Multiplying both sides by the deterministic quantity S_T^0 yields

$$E^Q[X] = E^{\tilde{Q}}[X] \tag{3.32}$$

and so

$$Q(A) = \tilde{Q}(A). \tag{3.33}$$

Hence, $Q = \tilde{Q}$, since $A \in \mathcal{F}_T = \mathcal{F}$ was arbitrary.

Conversely, suppose the market is viable but *not* complete. Then we will show that there is more than one equivalent martingale measure. Since the market is not complete, there exists a European contingent claim X that is not attainable. Let $\mathcal{P}$ denote the set of $\hat{\phi} = (\hat{\phi}^1, \ldots, \hat{\phi}^d)$ satisfying $\hat{\phi}^i = \{\hat{\phi}_t^i, t = 1, \ldots, T\}$ where $\hat{\phi}_t^i$ is a real-valued $\mathcal{F}_{t-1}$-measurable random variable for $i = 1, \ldots, d$, $t = 1, \ldots, T$. We claim that there is no pair $(c, \hat{\phi})$ such that $\hat{\phi} \in \mathcal{P}$, $c \in \mathbb{R}$ and

$$c + \sum_{t=1}^{T} \hat{\phi}_t \cdot \Delta \hat{S}_t^* = X^*, \tag{3.34}$$

where $\hat{S}^* = (S^{*,1}, \ldots, S^{*,d})$. To see this, observe that if there were such a pair $(c, \hat{\phi})$, then by Lemma 3.2.5, $\hat{\phi}$ could be extended to a trading strategy $\phi = (\phi^0, \hat{\phi}^1, \ldots, \hat{\phi}^d)$ with initial value cS_0^0, and then, by (3.11)–(3.12), ϕ would be a replicating strategy for X, which yields a contradiction.

Now, adopting the same device as in the proof of Theorem 3.2.4 of viewing random variables as vectors in $\mathbb{R}^n$, let

$$L = \left\{ c + \sum_{t=1}^{T} \hat{\phi}_t \cdot \Delta \hat{S}_t^* : \hat{\phi} \in \mathcal{P},\ c \in \mathbb{R} \right\}. \tag{3.35}$$

Then, L is a linear subspace of $\mathbb{R}^n$ (since c is allowed to take any value in $\mathbb{R}$) and $X^* \notin L$. It follows that L is a strict subspace of $\mathbb{R}^n$ and there is a

vector $Z \in \mathbb{R}^n \setminus \{0\}$ such that $Z \in L^{\perp}$, the orthogonal complement of L in $\mathbb{R}^n$. Then,

$$\sum_{\omega\in\Omega} Z(\omega)Y(\omega) = 0 \qquad \text{for all } \ Y \in L. \tag{3.36}$$

Since the finite market model is viable, there is at least one equivalent martingale measure P^*. Then $P^*(\{\omega\}) > 0$ for all $\omega \in \Omega$, and on setting

$$\tilde{Z}(\omega) = \frac{Z(\omega)}{P^*(\omega)} \quad \text{for all } \omega \in \Omega, \tag{3.37}$$

we may rewrite (3.36) as

$$E^{P^*}[\tilde{Z}Y] = 0 \quad \text{for all } Y \in L. \tag{3.38}$$

Now, define

$$P^{**}(\{\omega\}) = \left(1 + \frac{\tilde{Z}(\omega)}{2\|\tilde{Z}\|_\infty}\right) P^*(\{\omega\}) \quad \text{for each } \omega \in \Omega, \tag{3.39}$$

where $\|\tilde{Z}\|_\infty = \max_{\omega\in\Omega} |\tilde{Z}(\omega)|$. Since $\tilde{Z} \neq 0$, $P^{**} \neq P^*$. Moreover, $P^{**}(\{\omega\}) > 0$ for each $\omega \in \Omega$. To check that P^{**} is a probability measure, note that

$$\begin{aligned} P^{**}(\Omega) &= P^*(\Omega) + \sum_{\omega\in\Omega} \frac{\tilde{Z}(\omega)}{2\,\|\tilde{Z}\|_\infty} P^*(\{\omega\}) \\ &= 1 + \frac{1}{2\,\|\tilde{Z}\|_\infty} E^{P^*}\left[\tilde{Z}\right] \\ &= 1, \end{aligned}$$

where we used (3.38) with $Y = 1 \in L$ to obtain the last line. Thus, P^{**} is a probability measure that is equivalent to P^*. We finally need to check that S^* is a martingale under P^{**}. For any $\hat{\phi} \in \mathcal{P}$,

$$\begin{aligned} E^{P^{**}}\left[\sum_{t=1}^{T} \hat{\phi}_t \cdot \Delta\hat{S}_t^*\right] &= E^{P^*}\left[\sum_{t=1}^{T} \hat{\phi}_t \cdot \Delta\hat{S}_t^*\right] \\ &\quad + \frac{1}{2\,\|\tilde{Z}\|_\infty} E^{P^*}\left[\tilde{Z}\sum_{t=1}^{T} \hat{\phi}_t \cdot \Delta\hat{S}_t^*\right]. \end{aligned}$$

The first term on the right side of the equality above is zero, by Lemma 3.2.6, since S^* is a martingale under P^*. The second term there is zero by (3.38), since $Y = \sum_{t=1}^{T} \hat{\phi}_t \cdot \Delta\hat{S}_t^* \in L$. On setting $\hat{\phi}_t^j = 0$ for all $j \neq i$ and $t = 1, 2, \ldots, T$, and applying Lemma 3.2.6 again, it follows that $S^{*,i}$ is a P^{**}-martingale for $i = 1, \ldots, d$; and since this is trivially so for $i = 0$, it follows that S^* is a P^{**} martingale and hence P^{**} is an equivalent martingale measure that is different from P^*. □

The following can be viewed as a form of the martingale representation theorem in a finite market model context. It gives an alternative characterization of completeness in a viable finite market model.

Theorem 3.3.3. *Suppose the finite market model is viable and P^* is an equivalent martingale measure. Then, the model is complete if and only if each real-valued martingale $M = \{M_t, \mathcal{F}_t, t = 0, 1, \ldots, T\}$ under P^* has a representation of the form*

$$M_t = M_0 + \sum_{s=1}^{t} \hat{\phi}_s \cdot \Delta \hat{S}^*_s, \qquad t = 0, 1, \ldots, T, \tag{3.40}$$

for some $\hat{\phi} = (\hat{\phi}^1, \ldots, \hat{\phi}^d)$ where for each $i = 1, \ldots, d$, $\hat{\phi}^i = \{\hat{\phi}^i_t, t = 1, \ldots, T\}$ and $\hat{\phi}^i_t$ is a real-valued $\mathcal{F}_{t-1}$-measurable random variable for each t. (Here $\hat{S}^* = (S^{*,1}, S^{*,2}, \ldots, S^{*,d})$.)

Proof. Suppose the model is complete and let $M = \{M_t, \mathcal{F}_t, t = 0, 1, \ldots, T\}$ be a martingale under P^*. Then, $X = M_T S^0_T$ is a European contingent claim. Since the model is complete, there exists a replicating strategy ϕ for X. Then, $V^*_T(\phi) = X^* = M_T$, and by Theorem 3.3.1 we have for $t = 0, 1, \ldots, T-1$,

$$V^*_t(\phi) = E^{P^*}[X^* \mid \mathcal{F}_t] = E^{P^*}[M_T \mid \mathcal{F}_t]. \tag{3.41}$$

Since M is a P^*-martingale, the last member above is equal to M_t P^*-a.s., and so it follows upon using (3.11)–(3.12) that for $t = 0, 1, \ldots, T$,

$$\begin{aligned} M_t = V^*_t(\phi) &= V^*_0(\phi) + G^*_t(\phi) \\ &= V^*_0(\phi) + \sum_{s=1}^{t} \phi_s \cdot \Delta S^*_s \\ &= M_0 + \sum_{s=1}^{t} \hat{\phi}_s \cdot \Delta \hat{S}^*_s \end{aligned}$$

where $\hat{\phi} = \{(\phi^1_t, \ldots, \phi^d_t), t = 1, \ldots, T\}$ and $\phi^i_t \in \mathcal{F}_{t-1}$ for $t = 1, \ldots, T$, $i = 1, \ldots, d$. Here we have used the fact that $G^*_t(\phi)$ involves only the holdings in the risky assets.

Conversely, suppose the representation property holds. To show that the model is complete, consider a European contingent claim X. Define

$$M_t = E^{P^*}[X^* \mid \mathcal{F}_t], \quad t = 0, 1, \ldots, T. \tag{3.42}$$

Then $M = \{M_t, \mathcal{F}_t, t = 0, 1, \ldots, T\}$ is a martingale under P^*. Let $\hat{\phi}$ be as in the representation (3.40). By Lemma 3.2.5, $\hat{\phi}$ can be extended to a

trading strategy $\phi = (\phi^0, \hat{\phi}^1, \ldots, \hat{\phi}^d)$ with initial value $M_0 S_0^0$. Then, for $t = 0, 1, \ldots, T$,

$$\begin{aligned} V_t^*(\phi) &= V_0^*(\phi) + G_t^*(\phi) \\ &= M_0 + \sum_{s=1}^{t} \hat{\phi}_s \cdot \Delta \hat{S}_s^* \\ &= M_t. \end{aligned}$$

Hence, $V_T^*(\phi) = M_T = X^*$, and it follows that $V_T(\phi) = X$. Thus, ϕ is a replicating strategy for X. Since X was an arbitrary European contingent claim, it follows that the market is complete. □

3.4. Pricing European Contingent Claims

In this section, we assume that the finite market model is viable and complete. Let P^* be the unique equivalent martingale measure. Consider a European contingent claim with value X at time T. Here we generalize from the situation treated in Chapter 2, where trading of the European contingent claim was allowed only at time zero, to allow it to be traded at all times $t = 0, 1, 2, \ldots, T-1$. (We leave it as an exercise for the reader to formulate the appropriate notion of arbitrage and to verify that the unique arbitrage free initial price remains valid when the European contingent claim can be bought or sold only at time zero.)

To determine the arbitrage free price process for the European contingent claim, we consider a market that allows trading in the stock, bond and European contingent claim at each time $t = 0, 1, \ldots, T-1$. Let $\{C_t, t = 0, 1, \ldots, T\}$ be an $\{\mathcal{F}_t\}$-adapted process, where C_t represents the price of the European contingent claim at time $t = 0, 1, \ldots, T-1$ and $C_T = X$. A *trading strategy* in the stocks, bond and the European contingent claim is a collection $\psi = \{(\phi_t, \gamma_t), t = 1, \ldots, T\}$ where for each t, $\phi_t = (\phi_t^0, \phi_t^1, \ldots, \phi_t^d)$ is a $(d+1)$-dimensional $\mathcal{F}_{t-1}$-measurable random vector such that for $i = 0, 1, \ldots, d$, ϕ_t^i represents the number of "shares" of asset i held over the time interval $(t-1, t]$, and γ_t is a real-valued $\mathcal{F}_{t-1}$-measurable random variable representing the number of European contingent claims held over the time interval $(t-1, t]$. This trading strategy must be self-financing; i.e., its initial value is

$$V_0(\psi) = \phi_1 \cdot S_0 + \gamma_1 C_0,$$

and at each time $t = 1, \ldots, T-1$,

$$\phi_t \cdot S_t + \gamma_t C_t = \phi_{t+1} \cdot S_t + \gamma_{t+1} C_t.$$

The value of the stocks-bond-contingent claim portfolio at time T is

$$V_T(\psi) = \phi_T \cdot S_T + \gamma_T X.$$

An *arbitrage opportunity* in the stocks-bond-contingent claim market is a trading strategy ψ such that $V_0(\psi) = 0$, $V_T(\psi) \geq 0$ and $E[V_T(\psi)] > 0$.

Remark. One may think of the stocks-bond-contingent claim market as a finite market model in its own right, with $d+1$ risky assets (d stocks and one contingent claim) and one riskless asset (one bond). Then the above notions of trading strategy and arbitrage coincide with those defined earlier in this chapter.

Theorem 3.4.1. *Suppose the finite market model is viable and complete and P^* is the unique equivalent martingale measure. Then for any European contingent claim X,*

$$\{S_t^0 E^{P^*}[X^* \mid \mathcal{F}_t], t = 0, 1, \ldots, T\}$$

is the (unique) arbitrage free price process for the European contingent claim, where $X^* = X/S_T^0$ is the discounted value of X at time T.

Remark. Here uniqueness of the price process is uniqueness up to indistinguishability of stochastic processes.

Proof. Let ϕ be a replicating strategy for X. Then by Theorem 3.3.1,

$$V_t(\phi) = S_t^0 E^{P^*}[X^* \mid \mathcal{F}_t], \quad t = 0, 1, \ldots, T.$$

Let $\{C_t, t = 0, 1, \ldots, T\}$ be the price process for the European contingent claim, where $C_T = X$.

We first show that if $P(C_s \neq V_s(\phi)) > 0$ for some s, then there is an arbitrage opportunity. Note $C_T = V_T(\phi) = X$. Suppose there is $s \in \{0, 1, \ldots, T-1\}$ such that $P(C_s > V_s(\phi)) > 0$. Let $A = \{\omega : C_s(\omega) > V_s(\phi)(\omega)\}$. Then $A \in \mathcal{F}_s$. An investor could act as follows to achieve an arbitrage. The investor invests nothing in stocks, bond or contingent claim up to time s. If $C_s \leq V_s(\phi)$, the investor continues to invest nothing from time s to T. If $C_s > V_s(\phi)$, at time s, the investor sells one European contingent claim, invests $V_s(\phi)$ of the proceeds in the market from time s onwards according to the strategy $(\phi_{s+1}, \ldots, \phi_T)$, and puts the remainder, $C_s - V_s(\phi)$, in the bond from time s to T. This strategy may be formally written as

$$\psi_t = \begin{cases} 0 & \text{for } t \leq s \\ \left(\phi_t^0 + \frac{C_s - V_s(\phi)}{S_s^0}, \phi_t^1, \ldots, \phi_t^d, -1\right) 1_A & \text{for } t > s. \end{cases}$$

It is readily verified to be self-financing. In particular, $V_t(\psi) = 0$ for $t \leq s$, and

$$1_A(\phi_{s+1} \cdot S_s + C_s - V_s(\phi) - C_s) = 0.$$

Now,

$$\begin{aligned} V_T(\psi) &= 1_A\left(\phi_T \cdot S_T + (C_s - V_s(\phi))\frac{S_T^0}{S_s^0} - X\right) \\ &= 1_A\left((C_s - V_s(\phi))\frac{S_T^0}{S_s^0}\right), \end{aligned}$$

since $V_T(\phi) = \phi_T \cdot S_T = X$, by the replicating property of ϕ. Thus, $V_T(\psi) \geq 0$ and $P(V_T(\psi) > 0) = P(A) > 0$, and ψ is an arbitrage.

Similarly, if $s \in \{0, 1, \dots, T-1\}$ such that $P(C_s < V_s(\phi)) > 0$ and $B = \{C_s < V_s(\phi)\}$, and we let

$$\psi_t = \begin{cases} 0 & \text{for } t \leq s \\ \left(-\phi_t^0 + \frac{V_s(\phi) - C_s}{S_s^0}, -\phi_t^1, \dots, -\phi_t^d, 1\right)1_B & \text{for } t > s, \end{cases}$$

for $t = 1, \dots, T$, then ψ represents an arbitrage. Thus we have shown that if $P(C_s \neq V_s(\phi)$ for some $s) > 0$, then there is an arbitrage opportunity.

It remains to show that if $C_s = V_s(\phi)$ for all s, there is no arbitrage opportunity in the stocks-bond-contingent claim market. For a contradiction, suppose $C_s = V_s(\phi)$ for all s and that $\psi = \{(\tilde{\phi}_t, \tilde{\gamma}_t), t = 1, \dots, T\}$ is an arbitrage opportunity. Then $V_0(\psi) = 0$, $V_T(\psi) \geq 0$ and $E[V_T(\psi)] > 0$. Let $V_t^*(\psi) = V_t(\psi)/S_t^0, t = 0, 1, \dots, T$. Then for $t \in \{1, \dots, T\}$,

$$\begin{aligned} E^{P^*}[V_t^*(\psi) \mid \mathcal{F}_{t-1}] &= E^{P^*}[\tilde{\phi}_t \cdot S_t^* + \tilde{\gamma}_t V_t^*(\phi) \mid \mathcal{F}_{t-1}] \\ &= \tilde{\phi}_t \cdot E^{P^*}[S_t^* \mid \mathcal{F}_{t-1}] + \tilde{\gamma}_t E^{P^*}[V_t^*(\phi) \mid \mathcal{F}_{t-1}] \\ &= \tilde{\phi}_t \cdot S_{t-1}^* + \tilde{\gamma}_t V_{t-1}^*(\phi) \end{aligned}$$

where for the last equality we have used the facts that S^* and $V^*(\phi)$ are martingales under P^* (cf. (3.29)). Using the self-financing property of ψ, we recognize the last expression above as $V_{t-1}^*(\psi)$. Hence, $\{V_t^*(\psi), \mathcal{F}_t, t = 0, 1, \dots, T\}$ is a P^*-martingale and so

$$E^{P^*}[V_T^*(\psi)] = E^{P^*}[V_0^*(\psi)]. \tag{3.43}$$

Now, $V_0^*(\psi) = V_0(\psi) = 0$, $V_T^*(\psi) \geq 0$, so it follows that $P^*(V_T^*(\psi) = 0) = 1$; and since P^* is equivalent to P, $P(V_T^*(\psi) = 0) = 1$. Hence, $P(V_T(\psi) = 0) = 1$, which contradicts the assumption that ψ is an arbitrage.

Thus, $C_t = V_t(\phi) = S_t^0 E^{P^*}[X^* \mid \mathcal{F}_t]$, $t = 0, 1, \dots, T$, defines the unique arbitrage free price process. □

Remark. Examination of the above proof shows that the conclusion of the theorem still holds if the finite market model is viable and X is simply replicable; i.e., one does not need to assume completeness of the market model.

3.5. Incomplete Markets

In this section we give a brief introduction to incomplete markets in the finite market model setting. For simplicity, we consider only a single time period; i.e., we assume that $T = 1$. However, many of the properties described here generalize to multiperiod models. We assume that the finite market model has no arbitrage opportunities; i.e., it is viable. It follows that there is at least one equivalent martingale measure.

Let X be a European contingent claim. A *superhedging strategy* for X is a trading strategy ϕ such that $V_T(\phi) \geq X$. By Theorem 3.3.1, the initial value of such a strategy is given by $E^{P^*}[Y^*]S_0^0$ where $Y^* = V_T(\phi)/S_T^0$ and P^* is any equivalent martingale measure. Thus,

$$\begin{aligned} V_+(X) &\equiv \inf\{V_0(\phi) : \phi \text{ is a superhedging strategy for } X\} \\ &= \inf\{E^{P^*}[Y^*]S_0^0 : Y \text{ is replicable and } Y \geq X\}, \end{aligned} \tag{3.44}$$

for any equivalent martingale measure P^*. From the last line it follows that $V_+(X) \geq E^{P^*}[X^*]S_0^0$ for each equivalent martingale measure P^*. The infimum in the definition of $V_+(X)$ is actually attained by some superhedging strategy ϕ. To see this, recall that Ω is a finite set and note that $V_+(X)$ is the optimal value of the linear program

$$\begin{aligned} \text{minimize} \quad & \phi_1 \cdot S_0 \\ \text{subject to} \quad & \phi_1 \cdot S_1 \geq X, \\ & \phi_1 \in \mathbb{R}^{d+1}. \end{aligned}$$

(Here we have used the fact that $T = 1$. In particular, $\phi_1 \in \mathcal{F}_0$ is a constant vector.) Since the feasible set for this linear program is easily seen to be nonempty and the objective function is bounded below (by $E^{P^*}[X^*]S_0^0$ where P^* is any equivalent martingale measure), it follows that the optimal value is attained (cf. Bertsimas and Tsitsiklis [**3**], page 67). A superhedging strategy for X whose initial value equals $V_+(X)$ is called a *minimal superhedging strategy*.

If the contingent claim is initially priced at $V_+(X)$, a seller of the claim can invest the proceeds of the sale in a minimal superhedging strategy ϕ and the seller is protected from the risk of having sold the contingent claim, since the seller's portfolio value at time T will be $V_T(\phi) - X \geq 0$. Furthermore, if the contingent claim is initially priced at more than $V_+(X)$, there is an arbitrage opportunity for the seller (the proof is left to the reader).

On the other hand, a buyer of the contingent claim wants the initial price C_0 of the contingent claim to be such that the buyer can invest $-C_0$ in a trading strategy ϕ with $V_0(\phi) = -C_0$ and such that the buyer is protected from the risk of having bought the contingent claim; i.e., $V_T(\phi) + X \geq 0$.

The supremum of such initial prices C_0 is

$$\begin{aligned} V_-(X) &\equiv \sup\{-V_0(\phi) : \phi \text{ is a trading strategy, } V_T(\phi) + X \geq 0\} \\ &= \sup\{V_0(\phi) : \phi \text{ is a trading strategy, } V_T(\phi) \leq X\} \\ &= \sup\{E^{P^*}[Y^*]S_0^0 : Y \text{ is replicable and } Y \leq X\}, \end{aligned}$$

for any equivalent martingale measure P^*. In the next to last line above we have used the fact that ϕ is a trading strategy if and only if $-\phi$ is a trading strategy and that the value of a trading strategy at each time t is a linear function of the trading strategy. In a similar manner to that for $V_+(X)$, one can see that the supremum in the definition of $V_-(X)$ is attained and $V_-(X) \leq E^{P^*}[X^*]S_0^0$ for all equivalent martingale measures P^*. Moreover, if the initial price of the contingent claim is priced less than $V_-(X)$, there is an arbitrage opportunity for the buyer of the contingent claim.

Let $\mathcal{M}$ denote the set of equivalent martingale measures. Then, from the above, we have

$$V_-(X) \leq \inf_{P^* \in \mathcal{M}} E^{P^*}[X^*]S_0^0 \leq \sup_{P^* \in \mathcal{M}} E^{P^*}[X^*]S_0^0 \leq V_+(X). \tag{3.45}$$

If X is replicable, then from the Remark at the end of Section 3.4, we have that $V_+(X) = V_-(X)$ is the unique arbitrage free initial price for the contingent claim X.

If X is not replicable, there are no arbitrage free initial prices outside the closed interval $[V_-(X), V_+(X)]$. In fact, $V_-(X)$ and $V_+(X)$ are also not arbitrage free initial prices in this case. For $V_+(X)$, this follows from the fact that the inf in the definition of $V_+(X)$ is attained and a minimal superhedging strategy ϕ must have $V_T(\phi) > X$ with positive probability, since otherwise ϕ would be a replicating strategy for X. A similar argument applies for $V_-(X)$. Using the definition of $V_-(X)$, $V_+(X)$, and the martingale property of S^* under any equivalent martingale measure, one can use an argument by contradiction to show that any initial price in the open interval $(V_-(X), V_+(X))$ is arbitrage free. This last interval is non-empty if X is not replicable, as the following lemma shows.

Lemma 3.5.1. *Suppose that a European contingent claim X is not replicable. Then $V_-(X) < V_+(X)$.*

Proof. Let P^* be an equivalent martingale measure. Recall from the proof of the second fundamental theorem of asset pricing, Theorem 3.3.2, that if X is not replicable, then X^* is not in

$$L = \{V_T^*(\phi) : \phi \text{ is a trading strategy}\}.$$

Decomposing X^* into its component X_L^* in L and its non-trivial component Z that is perpendicular to L, we have

$$X^* = X_L^* + Z. \tag{3.46}$$

From the proof of Theorem 3.3.2 we see that P^{**}, defined by

$$P^{**}(\{\omega\}) = \left(1 + \frac{\tilde{Z}(\omega)}{2\|\tilde{Z}\|_\infty}\right) P^*(\{\omega\}), \quad \omega \in \Omega, \tag{3.47}$$

where $\tilde{Z}$ is given by (3.37), is an equivalent martingale measure. From Theorem 3.3.1, since $X_L^* = V_T^*(\phi)$ for some trading strategy ϕ, we have $E^{P^*}[X_L^*] = E^{P^{**}}[X_L^*] = V_0^*(\phi)$. Now,

$$E^{P^{**}}[Z] = E^{P^*}[Z] + \frac{Z \cdot Z}{2\|\tilde{Z}\|_\infty} > E^{P^*}[Z], \tag{3.48}$$

since $Z \neq 0$. Hence, $E^{P^*}[X^*] < E^{P^{**}}[X^*]$, and so using (3.45) we have

$$V_-(X) \leq E^{P^*}[X^*]S_0^0 < E^{P^{**}}[X^*]S_0^0 \leq V_+(X). \tag{3.49}$$

□

The following is a consequence of Theorem 3.3.1 and the proof of the above theorem.

Corollary 3.5.2. *A European contingent claim X is replicable if and only if $E^{P^*}[X^*]$ has the same value for all equivalent martingale measures P^*.*

We also have the following alternative characterization of $V_+(X)$ and $V_-(X)$, which implies that the first and last inequalities in (3.45) are actually equalities.

Theorem 3.5.3. *For any European contingent claim X,*

$$\begin{aligned} V_+(X) &= \sup\left\{E^{P^*}[X^*]S_0^0 : P^* \in \mathcal{M}\right\}, \\ V_-(X) &= \inf\left\{E^{P^*}[X^*]S_0^0 : P^* \in \mathcal{M}\right\}. \end{aligned}$$

Proof. We prove the equality for $V_+(X)$; the proof for $V_-(X)$ is similar. Viewing probabilities on the finite set Ω as vectors in n-dimensional Euclidean space (where $n = |\Omega|$), let $P_1^*, \ldots, P_J^*$ be a maximal linearly independent subset of $\mathcal{M}$. Then any element of $\mathcal{M}$ can be written in the form $\sum_{j=1}^J \theta_j P_j^*$ for some real constants θ_j, $j = 1, \ldots, J$ satisfying $\sum_{j=1}^J \theta_j = 1$. It follows from Corollary 3.5.2 that a European contingent claim Y is replicable if and only if $E^{P^*}[Y^*]$ has the same value for all $P^* \in \mathcal{M}$, and this

occurs if and only if $E^{P_j^*}[Y^*]$ has the same value for $j = 1, \ldots, J$. Then, by (3.44), $V_+(X)$ is the optimal value of the linear program

$$\begin{array}{ll} \text{minimize} & \lambda \\ \text{subject to} & \lambda = E^{P_j^*}[Y^*]S_0^0, \; j = 1, \ldots, J, \\ & Y^* \geq X^*. \end{array}$$

The dual to this linear program is

$$\begin{array}{ll} \text{maximize} & \psi \cdot X^* \\ \text{subject to} & \psi = \sum_{j=1}^{J} \theta_j P_j^* S_0^0, \\ & \psi \geq 0, \; \sum_{j=1}^{J} \theta_j = 1, \\ & \psi \in \mathbb{R}^n, \; \theta \in \mathbb{R}^J. \end{array}$$

Let $\tilde{\mathcal{M}}$ denote the set of probabilities on Ω under which S^* is a martingale, and let

$$\mathcal{N} = \{\psi / S_0^0 : \psi \in \mathbb{R}^n \text{ is feasible for the above dual program}\}. \tag{3.50}$$

We claim that

$$\mathcal{N} = \tilde{\mathcal{M}} = \overline{\mathcal{M}}, \tag{3.51}$$

where $\overline{\mathcal{M}}$ denotes the closure of $\mathcal{M}$ in $\mathbb{R}^n$. This claim can be verified by noting the following.

(a) Each element of $\mathcal{M}$ when multiplied by S_0^0 yields a feasible ψ for the dual program, and the set of feasible ψ for the dual program is closed, being the linear projection of the closed set of feasible pairs (ψ, θ) for the dual program (cf. Bertsimas and Tsitsiklis [**3**], Corollary 2.4, p. 74). Hence, $\overline{\mathcal{M}}$ is contained in $\mathcal{N}$.

(b) For a feasible ψ for the dual program, ψ / S_0^0 is an element of $\tilde{\mathcal{M}}$. Although an element of $\tilde{\mathcal{M}}$ need not be strictly positive on all of Ω, it can be approximated arbitrarily closely by an element of $\mathcal{M}$, since there is at least one equivalent martingale measure. Hence, $\mathcal{N} \subset \tilde{\mathcal{M}} \subset \overline{\mathcal{M}}$.

Since the original linear program has a finite optimal value, the same is true of the dual program, and the optimal values for the two programs are the same and they are attained.

Combining the above, we obtain

$$V_+(X) \;=\; \max\left\{E^Q[X^*]S_0^0 : Q \in \tilde{\mathcal{M}}\right\} = \sup\left\{E^{P^*}[X^*]S_0^0 : P^* \in \mathcal{M}\right\}.$$

□

Now, by (3.45), Lemma 3.5.1 and Theorem 3.5.3, when X is not replicable,

$$V_-(X) = \inf_{P^* \in \mathcal{M}} E^{P^*}[X^*]S_0^0 < \sup_{P^* \in \mathcal{M}} E^{P^*}[X^*]S_0^0 = V_+(X), \tag{3.52}$$

and all of the prices in the non-empty interval $(V_-(X), V_+(X))$ are arbitrage free initial prices, and neither the buyer nor the seller can completely eliminate the risk of buying or selling the contingent claim for such a price. To further refine the choice of an initial arbitrage free price and a hedging strategy, various procedures have been proposed. For a summary of an approach based on utility maximization under the subjective probability P, see Davis [**13**].

For further discussion of incomplete markets in discrete time, see Föllmer and Schied [**16**] (Chapter 10) and Musiela and Rutkowski [**32**] (Chapter 4), for a start.

3.6. Separating Hyperplane Theorem

Theorem 3.6.1. *(Separating Hyperplane Theorem) Let F be a compact, convex, non-empty subset of $\mathbb{R}^n$. Let L be a non-empty linear subspace of $\mathbb{R}^n$. Suppose $F \cap L = \emptyset$. Then there exists a hyperplane*

$$H = \{x \in \mathbb{R}^n : z \cdot x = 0\} \text{ for some } z \in \mathbb{R}^n \setminus \{0\}$$

such that $L \subset H$ and $z \cdot x > 0$ for all $x \in F$.

Proof. Define $G = F - L = \{x \in \mathbb{R}^n : x = f - \ell \text{ for some } f \in F, \ell \in L\}$. Then G is convex, closed and non-empty. The convexity follows easily from that for F and L. To see that G is closed, consider a sequence $\{x_m = f_m - \ell_m\}_{m=1}^\infty$ in G where $f_m \in F$, $\ell_m \in L$ for all m. Suppose $x_m \to x \in \mathbb{R}^n$ as $m \to \infty$. Since F is compact, there is a subsequence $\{f_{m_r}\}_{r=1}^\infty$ converging to some $f \in F$. Then $\ell_{m_r} = -x_{m_r} + f_{m_r}$ converges to $-x + f$. Since L is closed we must have $-x + f \in L$. Hence $x = f - (-x + f) \in G$. Clearly G is non-empty, since F and L are both non-empty. Note that G does not contain the origin (otherwise F would intersect L non-trivially).

Let B denote the closed ball centered at the origin of radius $r > 0$ (where distance is measured by the Euclidean norm). Choose $r > 0$ such that $B \cap G \neq \emptyset$. Then $B \cap G$ is closed, bounded, non-empty and hence compact. So the continuous function $g(x) = \|x\|$ attains its infimum on $B \cap G$ at some $z \in B \cap G$, where $\|x\| = (x \cdot x)^{\frac{1}{2}}$ denotes the Euclidean norm of x. Note that $\|z\| \leq r$, since $z \in B$, and $z \neq 0$ since $0 \notin G$.

Now $\|x\| > r$ for $x \in G \setminus B$ and so combining this with the above we have $\|x\| \geq \|z\|$ for all $x \in G$. Then for any $\lambda \in (0,1)$ and $x \in G$, $\lambda x + (1-\lambda)z \in G$

by the convexity of G and so

$$\|\lambda x + (1-\lambda)z\|^2 \geq \|z\|^2 \text{ for all } x \in G, \lambda \in (0,1).$$

Expanding and dividing through by λ yields:

$$2(1-\lambda)x \cdot z - 2z \cdot z + \lambda(x \cdot x + z \cdot z) \geq 0.$$

Letting $\lambda \to 0$, we obtain

$$x \cdot z \geq z \cdot z \quad \text{ for all } x \in G.$$

Then, $(f - \ell) \cdot z \geq z \cdot z$ for all $f \in F$, $\ell \in L$, which implies

$$f \cdot z \geq \ell \cdot z + z \cdot z \quad \text{ for all } f \in F, \ell \in L.$$

Fix $f \in F$. Then

$$\ell \cdot z \leq f \cdot z - z \cdot z \quad \text{ for all } \ell \in L.$$

But L is a linear space, so the above holds with $\gamma\ell$ in place of ℓ for all $\gamma \in \mathbb{R}$. The only way this can be true is if $\ell \cdot z = 0$ for all $\ell \in L$. Hence, $f \cdot z \geq z \cdot z > 0$ for all $f \in F$.

Let $H = \{x \in \mathbb{R}^n : x \cdot z = 0\}$. Then from the above, $L \subset H$ and $f \cdot z > 0$ for all $f \in F$. □

3.7. Exercises

1. Consider the multi-period CRR binomial model introduced in Chapter 2. Assuming $u > d > 0$, verify that this model is viable if and only if $d < 1+r < u$. In this case, verify that the model is complete.

2. Let $T = 2$, $\Omega = \{\omega_1, \ldots, \omega_4\}$, $P(\{\omega_i\}) > 0$ for $i = 1, \ldots, 4$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \emptyset, \Omega\}$, and $\mathcal{F}_2$ be the collection of all subsets of Ω. Consider a riskless asset with price process $S^0 = \{S_t^0, t = 0, 1, 2\}$ where $S_t^0 = 1$ for all t, and a risky asset with price process $S^1 = \{S_t^1, t = 0, 1, 2\}$ such that

$$\begin{aligned}
S_0^1(\omega_1) &= 5, \quad S_1^1(\omega_1) = 8, \quad S_2^1(\omega_1) = 9 \\
S_0^1(\omega_2) &= 5, \quad S_1^1(\omega_2) = 8, \quad S_2^1(\omega_2) = 6 \\
S_0^1(\omega_3) &= 5, \quad S_1^1(\omega_3) = 4, \quad S_2^1(\omega_3) = 5 \\
S_0^1(\omega_4) &= 5, \quad S_1^1(\omega_4) = 4, \quad S_2^1(\omega_4) = 2.
\end{aligned}$$

Then

$$X = \max(0, S_0^1 - 6, S_1^1 - 6, S_2^1 - 6)$$

is the value at time T of a so-called *look-back* option, where this value depends on the maximum of the prices of the underlying asset S^1 over the time interval from 0 to T.

(a) Draw a tree to indicate the possible "paths" followed by the risky asset price process S^1.

(b) Find an equivalent martingale measure for the model.

(c) Find a replicating strategy for the option whose value at time T is given by X.

(d) What is the arbitrage free price for the option at time zero?

3. Consider a finite market model with $T = 2$, $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$, and $P(\{\omega_i\}) > 0$ for $i = 1, \ldots, 5$. Suppose there are two assets, a riskless asset with price process $S^0 = \{S_t^0, t = 0, 1, 2\}$ where $S_t^0 = (1+r)^t$ for $t = 0, 1, 2$, and some $r \geq 0$, and a risky asset with price process $S^1 = \{S_t^1, t = 0, 1, 2\}$ where

$$
\begin{aligned}
S_0^1(\omega_1) &= 5, \quad S_1^1(\omega_1) = 8, \quad S_2^1(\omega_1) = 9 \\
S_0^1(\omega_2) &= 5, \quad S_1^1(\omega_2) = 8, \quad S_2^1(\omega_2) = 7 \\
S_0^1(\omega_3) &= 5, \quad S_1^1(\omega_3) = 4, \quad S_2^1(\omega_3) = 6 \\
S_0^1(\omega_4) &= 5, \quad S_1^1(\omega_4) = 4, \quad S_2^1(\omega_4) = 5 \\
S_0^1(\omega_5) &= 5, \quad S_1^1(\omega_5) = 4, \quad S_2^1(\omega_5) = 2.
\end{aligned}
$$

Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \sigma\{S_0^1, S_1^1\}$ and $\mathcal{F}_2 = \sigma\{S_0^1, S_1^1, S_2^1\}$.

(a) Draw a tree to indicate the possible "paths" followed by the risky asset price process S^1.

(b) Suppose $r = 0.1$. Is there an equivalent martingale measure for this model? If there is one, is it unique? If there is not one, demonstrate an arbitrage opportunity. What are the answers to the last three questions if $r = 1$?

4. Let $T = 1$, $S_t^0 = (1+r)^t$ where $r = \frac{1}{3}$, $t = 0, 1$, $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, and $P(\{\omega_i\}) > 0$ for $i = 1, 2, 3, 4$. Suppose that there are two risky assets with price processes S^1, S^2:

		S_1^i			
i	S_0^i	ω_1	ω_2	ω_3	ω_4
1	2	4	4	8/3	4/3
2	5	8	16/3	16/3	8

(a) Specify the space

$$L = \{G_T^*(\phi) : \phi \text{ is a trading strategy with } V_0(\phi) = 0\}$$

for this example.

(b) Are there any arbitrage opportunities for this example? If so, find all of them.

(c) Are there any equivalent martingale measures for this example? If so, find all of them.

5. Show that the separating hyperplane theorem follows from the Hahn-Banach theorem.

Chapter 4

Black-Scholes Model

In this chapter we consider a simple continuous (in both time and space) financial market model called the Black-Scholes model. This can be viewed as a continuous analogue of the binomial model. Indeed, it can be obtained as a limit of a sequence of binomial models (with appropriate parameters). We shall prove the famous Black-Scholes formula for pricing European call options. In 1973, Fischer Black and Myron Scholes [**5**] developed a formula for the valuation of European contingent claims based on a geometric Brownian motion model for the stock price process. Robert Merton [**31**] developed another method to derive the formula that turned out to have very wide applicability; he also generalized the formula in many directions. In 1997, this work was recognized with the award of the Nobel Prize in Economics to Myron Scholes and Robert Merton (Fischer Black had passed away in 1995).

We begin this chapter by defining the Black-Scholes model and the notion of admissible trading strategy. We derive the equivalent martingale measure for this model and show that every suitably integrable European contingent claim has a replicating strategy. We use this to find the arbitrage free price process for any suitably integrable European contingent claim. We apply this result to the pricing of a European call option, which results in the famous Black-Scholes option pricing formula. Using the solution of the Black-Scholes partial differential equation, we also exhibit a replicating strategy in this case. We then consider American contingent claims and derive the arbitrage free initial price for a suitably integrable American contingent claim. We illustrate this result for American call and put options.

In this and subsequent chapters, we shall use various notions and concepts from the theory of continuous time stochastic processes and stochastic integration. For the convenience of the reader, a summary of some relevant concepts and results is provided in Appendices C and D. The reader may find it helpful to briefly review the material in that appendix before proceeding further. For further details, the reader is encouraged to consult other texts such as Chung [**9**] and Chung and Williams [**11**], Karatzas and Shreve [**27**], Revuz and Yor [**35**], or the two-volume set by Rogers and D. Williams [**37, 36**].

Regarding notational conventions, in this and subsequent chapters, we continue to use those conventions mentioned at the beginning of Chapter 2. In particular, if X is a real-valued random variable defined on a probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$, we shall sometimes use the notation $X \in \mathcal{G}$ to indicate that X is $\mathcal{G}$-measurable. In contrast to the situation for the discrete models considered in previous chapters where there were only trivial sets of probability zero, from here on, typically there can be many sets of probability zero. (In)equalities between two random variables will be considered to hold almost surely. This will sometimes be indicated explicitly for emphasis, but often we shall omit special mention of the almost sure nature of (in)equalities. In a similar vein, a random variable will be called bounded if it is bounded almost surely. Two stochastic processes will be considered equal if they are indistinguishable (cf. Appendices B and C). In the following, log denotes the natural logarithm and for a vector $x \in \mathbb{R}^d$, $|x|$ will denote the Euclidean norm of x given by $|x| = \left(\sum_{i=1}^d (x^i)^2\right)^{\frac{1}{2}}$.

4.1. Preliminaries

We consider a finite time interval $[0, T]$, for some $0 < T < \infty$, as the interval during which trading may take place. The Borel σ-algebra of subsets of $[0, T]$ will be denoted by $\mathcal{B}_T$.

We assume as given a complete probability space $(\Omega, \mathcal{F}, P)$ on which is defined a standard one-dimensional *Brownian motion* $W = \{W_t, t \in [0, T]\}$. Let $\{\mathcal{F}_t, t \in [0, T]\}$ denote the standard filtration generated by the Brownian motion W under P (cf. Appendix D, Section D.1). It is well known that this filtration is right continuous; i.e., for each $t \in [0, T)$, $\mathcal{F}_t = \mathcal{F}_{t+} \equiv \cap_{s \in (t,T]} \mathcal{F}_s$ (cf. Chung [**9**], Section 2.3, Theorem 4). All random variables considered in this chapter will be assumed to be defined on $(\Omega, \mathcal{F}_T)$, and so without loss of generality we assume that $\mathcal{F} = \mathcal{F}_T$. We shall frequently write $\{\mathcal{F}_t\}$ instead of the more cumbersome $\{\mathcal{F}_t, t \in [0, T]\}$. The filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ satisfies the usual conditions (cf. Appendix C). It is well known that under this assumption, every martingale has a modification

whose paths are all right continuous with finite left limits; cf. Chung [**9**], Theorem 3, page 29 and Corollary 1, page 26. In fact, since the filtration is generated by the Brownian motion W, every martingale has a continuous modification; cf. Theorems II.2.9, V.3.5 in Revuz and Yor [**35**]. Expectations with respect to P will be denoted by $E[\cdot]$ unless there is more than one probability measure under consideration, in which case we shall use $E^P[\cdot]$ in place of $E[\cdot]$.

4.2. Black-Scholes Model

Our Black-Scholes model has two assets, a (risky) stock with price process $S = \{S_t, t \in [0,T]\}$ and a (riskless) bond with price process $B = \{B_t, t \in [0,T]\}$. These processes are given by

$$\begin{aligned} S_t &= S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right), \quad t \in [0,T], && (4.1) \\ B_t &= e^{rt}, \quad t \in [0,T], && (4.2) \end{aligned}$$

where $r \geq 0$, $\mu \in \mathbb{R}$, $\sigma > 0$, and $S_0 > 0$ is a positive constant. The parameter σ is called the *volatility*. The process S is frequently called a *geometric Brownian motion*. It is well known (cf. Chung and Williams [**11**], Theorem 6.2) that $\{e^{-\mu t}S_t, \mathcal{F}_t, t \in [0,T]\}$ is an L^2-martingale under P. The processes S, B are continuous, adapted, and satisfy the following dynamic equations P-a.s.:

$$\begin{aligned} S_t &= S_0 + \mu \int_0^t S_s\,ds + \sigma \int_0^t S_s\,dW_s, \quad t \in [0,T], && (4.3) \\ B_t &= B_0 + r\int_0^t B_s\,ds, \quad t \in [0,T]. && (4.4) \end{aligned}$$

We shall sometimes write these equations in the following differential form for convenience in performing calculations using Itô's formula (cf. Appendix D, Section D.4):

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t\,dW_t, \quad t \in [0,T], && (4.5) \\ dB_t &= rB_t dt, \quad t \in [0,T]. && (4.6) \end{aligned}$$

Remark. To see the connection with the binomial model, note that in the Black-Scholes model, $\log(S_t/S_0) = \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t$, a Brownian motion with drift, whereas in the binomial model, $\log(S_t/S_0) = \sum_{i=1}^t \xi_i$, which is a (possibly biased) random walk. It is well known that a sequence of random walks with appropriate rescaling and parameter values can be approximated by a Brownian motion with drift.

A *trading strategy* is a two-dimensional stochastic process $\phi = \{\phi_t = (\alpha_t, \beta_t), t \in [0,T]\}$ satisfying

(i) $\phi : [0,T] \times \Omega \to \mathbb{R}^2$ is $(\mathcal{B}_T \times \mathcal{F}_T)$-measurable where $\phi(t,\omega) = \phi_t(\omega)$ for each $t \in [0,T]$ and $\omega \in \Omega$;

(ii) ϕ is adapted, i.e., $\phi_t \in \mathcal{F}_t$ for each $t \in [0,T]$;

(iii) $\int_0^T \alpha_t^2 \, dt < \infty$ and $\int_0^T |\beta_t| \, dt < \infty$ P-a.s.

We sometimes abbreviate the description of conditions like (i) by saying that ϕ is $(\mathcal{B}_T \times \mathcal{F}_T)$-measurable. The above conditions ensure that integrals of the form

$$\begin{aligned}\int_0^t \alpha_s \, dS_s &= \mu \int_0^t \alpha_s S_s \, ds + \sigma \int_0^t \alpha_s S_s \, dW_s, \\ \int_0^t \beta_s \, dB_s &= r \int_0^t \beta_s B_s \, ds\end{aligned}$$

are finite P-a.s. By defining the integrals to be identically zero on the exceptional null set, we may and do assume that they define continuous, adapted stochastic processes (see the definition of an Itô process in Appendix D). Here α_t is interpreted as the number of shares of stock held at time t, and β_t is interpreted as the number of units of the bond held at time t. The *value* at time t of the portfolio associated with ϕ is given by

$$V_t(\phi) = \alpha_t S_t + \beta_t B_t. \tag{4.7}$$

A trading strategy ϕ is said to be *self-financing* if $V(\phi) = \{V_t(\phi), t \in [0,T]\}$ is a continuous adapted process such that P-a.s.,

$$V_t(\phi) = V_0(\phi) + \int_0^t \alpha_s dS_s + \int_0^t \beta_s dB_s, \quad \text{for all } t \in [0,T], \tag{4.8}$$

or, in other words,

$$dV_t(\phi) = \alpha_t dS_t + \beta_t dB_t, \qquad t \in [0,T]. \tag{4.9}$$

Thus, changes in the value of the portfolio result only from changes in the values of the assets so that there is no external infusion of capital and no spending of wealth.

Let ϕ be a trading strategy. The *discounted stock price process* and *discounted value process* associated with ϕ are given by

$$S_t^* = \frac{S_t}{B_t} = e^{-rt} S_t, \quad t \in [0,T], \tag{4.10}$$

$$V_t^*(\phi) = \frac{V_t(\phi)}{B_t} = e^{-rt} V_t(\phi), \quad t \in [0,T], \tag{4.11}$$

respectively. The pair (S, B^{-1}) is a two-dimensional Itô process where $B_t^{-1} = e^{-rt}$ is of bounded variation on $[0,T]$ and satisfies

$$dB_t^{-1} = -r B_t^{-1} \, dt, \quad t \in [0,T]. \tag{4.12}$$

By applying Itô's formula (cf. Appendix D, Section D.4) to this Itô process with the function $f(x,y) = xy$ for $x, y \in \mathbb{R}$, and using (4.5), (4.12), we have P-a.s. for all $t \in [0,T]$:

$$\begin{aligned} S_t^* &= S_0^* - r\int_0^t S_s^*\,ds + \int_0^t e^{-rs}\,dS_s \\ &= S_0^* + (\mu - r)\int_0^t S_s^*\,ds + \sigma\int_0^t S_s^*\,dW_s. \end{aligned} \tag{4.13}$$

Lemma 4.2.1. *A trading strategy ϕ is self-financing if and only if $V^*(\phi)$ is a continuous, adapted process such that P-a.s., for each $t \in [0,T]$,*

$$V_t^*(\phi) = V_0(\phi) + \int_0^t \alpha_s\,dS_s^*. \tag{4.14}$$

Here $\int_0^t \alpha_s\,dS_s^* = (\mu - r)\int_0^t \alpha_s\,S_s^*\,ds + \sigma\int_0^t \alpha_s\,S_s^*\,dW_s$.

Proof. In the following, for convenience, we use the differential forms of integral equations which hold P-a.s. However, each of the steps could be written in integral form to give a totally rigorous proof. The essential aspect is that we use the rules of stochastic calculus for manipulating differentials at each stage.

Note that $V(\phi)$ is continuous and adapted if and only if $V^*(\phi)$ is continuous and adapted. So it suffices to establish the equivalence of (4.14) with (4.8). Suppose that ϕ is a self-financing trading strategy. Then, by applying Itô's formula to the two-dimensional Itô process $(V(\phi), B^{-1})$ with the function $f(x,y) = xy$ for $x, y \in \mathbb{R}$, on using (4.11), (4.7), (4.9), and (4.12), we obtain P-a.s. for each $t \in [0,T]$,

$$\begin{aligned} dV_t^*(\phi) &= -re^{-rt}V_t(\phi)\,dt + e^{-rt}\,dV_t(\phi) \\ &= e^{-rt}\left(-r\alpha_t S_t\,dt - r\beta_t B_t\,dt + \alpha_t dS_t + \beta_t\,dB_t\right). \end{aligned}$$

Using (4.5), (4.6), and (4.13), the above yields

$$dV_t^*(\phi) = e^{-rt}\alpha_t\left((\mu - r)S_t\,dt + \sigma S_t\,dW_t\right) = \alpha_t\,dS_t^*, \tag{4.15}$$

for each $t \in [0,T]$, which proves that (4.14) holds.

Conversely, suppose that (4.14) holds. Then using (4.13), (4.6), and (4.7), we have P-a.s. for each $t \in [0,T]$,

$$\begin{aligned} dV_t^*(\phi) &= \alpha_t\,dS_t^* = e^{-rt}\alpha_t\left(-rS_t\,dt + dS_t\right) \\ &= e^{-rt}\left(-r\alpha_t S_t\,dt - r\beta_t B_t\,dt + \beta_t\,dB_t + \alpha_t\,dS_t\right) \\ &= e^{-rt}\left(-rV_t(\phi)\,dt + \alpha_t\,dS_t + \beta_t\,dB_t\right). \end{aligned}$$

Then, by applying Itô's formula to the Itô process $(V^*(\phi), B)$ and the function $f(x, y) = xy$ for $x, y \in \mathbb{R}$, we obtain

$$\begin{aligned} dV_t(\phi) &= d(e^{rt}V_t^*(\phi)) = re^{rt}V_t^*(\phi)\,dt + e^{rt}\,dV_t^*(\phi) \\ &= rV_t(\phi)\,dt + (-rV_t(\phi)\,dt + \alpha_t\,dS_t + \beta_t\,dB_t) \\ &= \alpha_t\,dS_t + \beta_t\,dB_t, \end{aligned}$$

and hence ϕ is self-financing. □

A self-financing trading strategy ϕ is an *arbitrage opportunity* if

$$V_0(\phi) = 0, \qquad V_T(\phi) \geq 0, \qquad \text{and} \qquad P\left(V_T(\phi) > 0\right) > 0. \tag{4.16}$$

An equivalent definition is obtained if the last term above is replaced by

$$E\left[V_T(\phi)\right] > 0,$$

since in the presence of the other conditions this is equivalent to $P(V_T(\phi) > 0) > 0$.

In contrast to the discrete time case, where we had only finitely many trading times and gains or losses are controlled by the initial investment, in continuous time, one can change one's strategy infinitely many times in a finite time interval and thereby obtain unbounded gains or losses by use of a doubling type of strategy. The following is a concrete example of such a strategy taken from Karatzas and Shreve [**28**].

Example. Consider the Black-Scholes model with $r = 0$, $\mu = 0$, and $\sigma = 1$. Then

$$B_t = 1 \qquad \text{and} \qquad dS_t = S_t dW_t, \qquad t \in [0, T]. \tag{4.17}$$

Define

$$I(t) = \int_0^t \frac{1}{\sqrt{T-s}}\,dW_s, \qquad \text{for all } 0 \leq t < T. \tag{4.18}$$

This defines a continuous, adapted stochastic integral process on $[0, T)$. The quadratic variation of I is given by

$$[I]_t = \int_0^t \frac{1}{T-s}ds = \log\left(\frac{T}{T-t}\right), \qquad \text{for each } t < T. \tag{4.19}$$

Note that $[I]_0 = 0$, $[I]_t$ is increasing with t, and $[I]_t \to \infty$ as $t \to T$. It follows that I can be time changed to a Brownian motion (cf. Chung and Williams [**11**], Section 9.3) and in particular, for each $a > 0$ and

$$\tau_a \equiv \inf\{t \in [0, T] : I(t) = a\} \wedge T, \tag{4.20}$$

we have $0 < \tau_a < T$ P-a.s. Fix $a > 0$ and let N denote the P-null set on which $\tau_a \geq T$. Define $\phi = \{(\alpha_t, \beta_t), t \in [0,T]\}$ by setting for each $t \in [0,T]$:

$$\alpha_t = \begin{cases} (T-t)^{-\frac{1}{2}} S_t^{-1} 1_{\{t \leq \tau_a\}}, & \text{on } N, \\ 0, & \text{on } N^c, \end{cases} \tag{4.21}$$

$$\beta_t = \begin{cases} I(t \wedge \tau_a) - \alpha_t S_t, & \text{on } N, \\ 0, & \text{on } N^c. \end{cases} \tag{4.22}$$

As $\mathcal{F}_t$ contains the P-null sets for each $t \in [0,T]$, it is straightforward to verify that ϕ satisfies the properties required of a trading strategy. Since $B_t = 1$ for all t, the value process for ϕ satisfies the following P-a.s. for all $t \in [0,T]$:

$$V_t(\phi) = I(t \wedge \tau_a) = \int_0^{t \wedge \tau_a} \frac{1}{\sqrt{T-s}}\, dW_s \tag{4.23}$$

$$= \int_0^t \alpha_s S_s\, dW_s = \int_0^t \alpha_s\, dS_s, \tag{4.24}$$

where we have used the fact that S solves (4.5) with $\mu = 0$ and $\sigma = 1$. Since $r = 0$, it follows that (4.14) holds and hence ϕ is a self-financing trading strategy. Note that $V_0(\phi) = 0$ and P-a.s., $V_T(\phi) = I(\tau_a) = a > 0$ and so ϕ is an arbitrage opportunity.

There are several ways to add conditions to ϕ to rule out such arbitrage strategies. These usually amount to constraints on the size of integrals of ϕ with respect to (S, B). Since pricing of contingent claims based on this model will use an equivalent martingale measure (or risk neutral probability), these conditions are often phrased in terms of such a probability measure. Some authors require that the associated value process is L^2-bounded under an equivalent martingale measure, some require that the discounted value process is a martingale under such a measure, and some put integrability constraints directly on ϕ. Before discussing our conditions on ϕ, we introduce the equivalent martingale measure.

4.3. Equivalent Martingale Measure

Definition 4.3.1. *Two probability measures, Q and $\tilde{Q}$, defined on $(\Omega, \mathcal{F})$ are equivalent (or mutually absolutely continuous) provided for each $A \in \mathcal{F}$,*

$$Q(A) = 0 \quad \text{if and only if} \quad \tilde{Q}(A) = 0. \tag{4.25}$$

An equivalent martingale measure (abbreviated as EMM) is a probability measure P^* on $(\Omega, \mathcal{F})$ such that P^* is equivalent to P and $\{S_t^*, \mathcal{F}_t, t \in [0,T]\}$ is a martingale under P^*.

Remark. An equivalent martingale measure is sometimes also called a *risk neutral probability.* In this chapter, we shall use the terms interchangeably.

We wish to show that such an equivalent martingale measure P^* exists. For this note that

$$S_t^* = S_0 \exp\left(\left(\mu - r - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right) \tag{4.26}$$

and in particular, P-a.s.,

$$dS_t^* = (\mu - r) S_t^* dt + \sigma S_t^* \, dW_t, \qquad t \in [0, T]. \tag{4.27}$$

The second term on the right in (4.27), when integrated, defines a martingale under P. We seek a probability measure P^* equivalent to P such that the discounted stock price process S^* satisfies a differential equation like (4.27) but without the drift term $(\mu - r) S_t^* dt$. For this, we use a simple form of the Girsanov transformation for changing the drift of a Brownian motion (cf. Appendix D, Theorem D.5.1). Let

$$\theta = \frac{\mu - r}{\sigma}, \tag{4.28}$$

$$\Lambda_t = \exp\left(-\theta W_t - \frac{1}{2}\theta^2 t\right), \qquad t \in [0, T]. \tag{4.29}$$

Then it is well known (cf. [**11**], Theorem 6.2) that $\{\Lambda_t, \mathcal{F}_t, t \in [0, T]\}$ is a positive martingale under P. On $(\Omega, \mathcal{F})$, we define a new probability measure P^* so that

$$\frac{dP^*}{dP} = \Lambda_T \qquad \text{on } \mathcal{F}; \tag{4.30}$$

i.e.,

$$P^*(A) = E^P\left[1_A \Lambda_T\right], \qquad A \in \mathcal{F}, \tag{4.31}$$

where we have added the superscript P to the expectation E to emphasize that it is computed under the probability measure P. Similarly, we shall use E^{P^*} to denote the expectation operator under P^*. Note that since $\{\Lambda_t, \mathcal{F}_t, t \in [0, T]\}$ is a P-martingale, $P^*(\Omega) = E^P[\Lambda_T] = \Lambda_0 = 1$ and so P^* is indeed a probability measure on $(\Omega, \mathcal{F})$. Since $\Lambda_T > 0$, it follows that for each $A \in \mathcal{F}$, $P^*(A) = 0$ if and only if $P(A) = 0$. Thus, P^* is equivalent to P. All that remains is to show that the discounted stock price process S^* is a martingale under P^*. For this, let

$$\tilde{W}_t = W_t + \theta t, \qquad t \in [0, T]. \tag{4.32}$$

By Girsanov's theorem (cf. Theorem D.5.1), $\{\tilde{W}_t, \mathcal{F}_t, t \in [0, T]\}$ is a standard Brownian motion and it is a martingale under P^*. Thus, for each $t \in [0, T]$, by (4.26) and (4.28), we have

$$\begin{aligned} S_t^* &= S_0 \exp\left(\left(\mu - r - \frac{1}{2}\sigma^2\right) t + \sigma\left(\tilde{W}_t - \theta t\right)\right) \\ &= S_0 \exp\left(-\frac{1}{2}\sigma^2 t + \sigma \tilde{W}_t\right), \end{aligned} \tag{4.33}$$

and so P^*-a.s.,

$$dS_t^* = \sigma S_t^* d\tilde{W}_t, \quad t \in [0, T], \tag{4.34}$$

where $\tilde{W}$ is a standard Brownian motion martingale under P^*. It is well known that the form (4.33) is that of an L^2-martingale with respect to $\{\mathcal{F}_t\}$, under P^* (cf. [**11**], Theorem 6.2). Thus, P^* is an equivalent martingale measure. In fact, P^* is the unique equivalent martingale measure.

An *admissible strategy* is a self-financing trading strategy ϕ such that $\{V_t^*(\phi), \mathcal{F}_t, t \in [0, T]\}$ is a martingale under the equivalent martingale measure P^*.

Given the stochastic differential equation (4.34) satisfied by S^* under P^* and the form (4.14) of $V^*(\phi)$, a sufficient condition for a self-financing trading strategy ϕ to be admissible is that

$$E^{P^*}\left[\int_0^T |\alpha_s S_s^*|^2 \, ds\right] < \infty. \tag{4.35}$$

In this case, the discounted value process will be an L^2-martingale under P^* (cf. Appendix D). The condition (4.35) holds if α is a bounded process.

4.4. European Contingent Claims

A *European contingent claim* is represented by an $\mathcal{F}_T$-measurable random variable X. A *replicating (or hedging) strategy* for a European contingent claim $X \in \mathcal{F}_T$ is an admissible strategy ϕ such that $V_T(\phi) = X$.

We shall use the following form of the *martingale representation theorem* for Brownian motion to obtain a replicating strategy for suitably integrable X. A more general form of this result is stated in Appendix D. A simpler version is stated here for convenience. For this theorem, it is important that a suitable filtration generated by a Brownian motion under the ambient probability measure is used. For this, note that $\tilde{W}$ is a standard one-dimensional Brownian motion under P^*. Now, since W and $\tilde{W}$ differ only by a deterministic process, it follows that $\sigma\{W_s : 0 \le s \le t\}$ is the same as $\sigma\{\tilde{W}_s : 0 \le s \le t\}$ for each $t \in [0, T]$. Furthermore, since P and P^* have the same null sets, augmenting by the P^*-null sets has the same effect as augmenting by the P-null sets. Consequently, the filtration $\{\mathcal{F}_t\}$ that was originally defined using W and augmentation by the P-null sets also satisfies $\mathcal{F}_t = \sigma\{\tilde{W}_s, s \in [0, t]\}^\sim$ for each $t \in [0, T]$, where the superscript $\sim$ denotes augmentation by the P^*-null sets. Consequently, $\{\mathcal{F}_t, t \in [0, T]\}$ is also the standard filtration generated by the Brownian motion $\tilde{W}$ on $(\Omega, \mathcal{F}, P^*)$. For a proof of the following martingale representation theorem, see Revuz and Yor [**35**], Theorem V.3.4.

Theorem 4.4.1. *(Martingale Representation Theorem) Suppose that* $M = \{M_t, \mathcal{F}_t, t \in [0,T]\}$ *is a right continuous martingale under* P^*. *Then there exists a* $(\mathcal{B}_T \times \mathcal{F}_T)$*-measurable, adapted process* $\eta = \{\eta_t, t \in [0,T]\}$ *such that* P^**-a.s.,* $\int_0^T \eta_s^2 \, ds < \infty$ *and*

$$M_t = M_0 + \int_0^t \eta_s \, d\tilde{W}_s \quad \text{for all } t \in [0,T]. \tag{4.36}$$

Theorem 4.4.2. *Suppose that* X *is an* $\mathcal{F}_T$*-measurable random variable such that* $E^{P^*}[|X|] < \infty$. *Then there exists a replicating strategy* ϕ *for* X. *Moreover, for any replicating strategy* ϕ *and* $X^* = X/B_T$, *we have that for each* $t \in [0,T]$, P^**-a.s.,*

$$V_t^*(\phi) = E^{P^*}[X^* \mid \mathcal{F}_t]. \tag{4.37}$$

Proof. We first prove that (4.37) holds if ϕ is a replicating strategy for X. Indeed, for such a ϕ, we have for fixed $t \in [0,T]$ that P^*-a.s.,

$$V_t^*(\phi) = E^{P^*}[V_T^*(\phi) \mid \mathcal{F}_t] = E^{P^*}[X^* \mid \mathcal{F}_t], \tag{4.38}$$

where the first equality follows from the admissibility of ϕ, which implies that $\{V_t^*(\phi), \mathcal{F}_t, t \in [0,T]\}$ is a martingale under P^*, and the second equality follows from the replicating property of ϕ: $V_T(\phi) = X$.

In light of the above, to prove the existence of a replicating strategy ϕ, it is natural to consider

$$M_t = E^{P^*}[X^* \mid \mathcal{F}_t], \qquad t \in [0,T]. \tag{4.39}$$

Note that $\{M_t, \mathcal{F}_t, t \in [0,T]\}$ is a P^*-martingale and hence it has a right continuous modification. In fact, since the filtration is generated by Brownian motion, M has a continuous modification (cf. Revuz and Yor [**35**], Theorem V.3.5). We again denote such a continuous modification by M. It follows from the martingale representation theorem, Theorem 4.4.1, that there is an adapted process $\eta = \{\eta_t, t \in [0,T]\}$ satisfying the conditions in Theorem 4.4.1 and such that P^*-a.s. for each $t \in [0,T]$,

$$M_t = M_0 + \int_0^t \eta_s \, d\tilde{W}_s. \tag{4.40}$$

Define

$$\alpha_t = \frac{\eta_t}{\sigma S_t^*}, \qquad t \in [0,T]. \tag{4.41}$$

Then $\alpha = \{\alpha_t, t \in [0,T]\}$ is $(\mathcal{B}_T \times \mathcal{F}_T)$-measurable, adapted and

$$\int_0^T \alpha_t^2 \, dt \le \left(\sigma \inf_{t \in [0,T]} S_t^*\right)^{-2} \int_0^T \eta_t^2 \, dt. \tag{4.42}$$

Note that $S_t^* > 0$ for each $t \in [0,T]$, and S_t^* is a continuous function of $t \in [0,T]$. Therefore, $\inf_{t\in[0,T]} S_t^* > 0$. So it follows from the integrability property of η that P^*-a.s.,

$$\int_0^T \alpha_t^2\, dt < \infty. \tag{4.43}$$

Define

$$\beta_t = M_t - \alpha_t S_t^*, \qquad t \in [0,T]. \tag{4.44}$$

Then $\beta = \{\beta_t, t \in [0,T]\}$ is $(\mathcal{B}_T \times \mathcal{F}_T)$-measurable and adapted, and using the Cauchy-Schwarz inequality we have

$$\begin{aligned}\int_0^T |\beta_t|\, dt &\leq \int_0^T |M_t|\, dt + \int_0^T |\alpha_t| S_t^*\, dt \\ &\leq T \max_{t\in[0,T]} |M_t| + \left(\max_{t\in[0,T]} S_t^*\right)\left(\int_0^T \alpha_t^2\, dt\right)^{\frac{1}{2}} T^{\frac{1}{2}}.\end{aligned}$$

The last line above is finite P^*-a.s. by (4.43) and the fact that M and S^* are continuous processes. Let $\phi = \{(\alpha_t, \beta_t), t \in [0,T]\}$. Then,

$$V_t^*(\phi) = \alpha_t S_t^* + \beta_t = M_t, \qquad t \in [0,T]. \tag{4.45}$$

This, together with (4.36), the definition of α, and (4.34), gives P^*-a.s.,

$$\begin{aligned}V_t^*(\phi) &= V_0(\phi) + \int_0^t \eta_s\, d\tilde{W}_s \\ &= V_0(\phi) + \int_0^t \alpha_s \sigma S_s^*\, d\tilde{W}_s \\ &= V_0(\phi) + \int_0^t \alpha_s\, dS_s^*,\end{aligned}$$

for each $t \in [0,T]$. Therefore, since P and P^* have the same null sets and $V^*(\phi) = M$ is continuous and adapted, ϕ is a self-financing trading strategy by Lemma 4.2.1. Combining this with (4.45) and the martingale property of M, we see that ϕ is admissible. Finally, taking $t = T$ in (4.45) implies that ϕ replicates X since $M_T = X^*$. □

4.5. Pricing European Contingent Claims

Consider a European contingent claim $X \in \mathcal{F}_T$ such that $E^{P^*}[|X|] < \infty$. To determine the arbitrage free price process for this claim, we need a notion of admissible strategy for trading in stock, bond and the contingent claim.

We assume that any given price process $C = \{C_t, t \in [0,T]\}$ for the European contingent claim is a right continuous, adapted process, where C_t represents the price of the European contingent claim at time t. (Adaptedness

is the minimal assumption we would expect on this process, and right continuity is a reasonable regularity assumption.) We suppose that for a given price process C, there is a set of admissible strategies Ψ_C for trading in stock, bond and the contingent claim. At a minimum, an element ψ of Ψ_C should be a three-dimensional adapted process $\psi = \{\psi_t = (\alpha_t, \beta_t, \gamma_t), t \in [0,T]\}$. We shall also want elements of Ψ_C to be self-financing in an appropriate way.

To show uniqueness of the price process, we do not want to place unnecessary restrictions on C just so that it can be used as an integrator (in defining self-financing trading strategies). In fact, to show uniqueness, we shall need only the reasonable assumption that rather simple strategies of the following type are admissible: those where one buys or sells one contingent claim, or does nothing, at some deterministic time t^* and then holds that position in the claim until time T, but where one may trade as usual in the stock and bond over $[t^*, T]$. More precisely, we assume that strategies of the following form are in Ψ_C:

$$\psi_t(\omega) = 1_A(\omega)(\tilde{\alpha}_t(\omega), \tilde{\beta}_t(\omega) + \tilde{\delta}_t(\omega), \tilde{\gamma}_t(\omega)), \quad \text{for } t \in [t^*, T], \tag{4.46}$$

and $\psi_t(\omega) = 0$ for $t \in [0, t^*)$, for all $\omega \in \Omega$, where t^* is a fixed time in $[0,T]$, $A \in \mathcal{F}_{t^*}$, $\{(\tilde{\alpha}_t, \tilde{\beta}_t), t \in [0,T]\}$ is an admissible self-financing trading strategy in the primary (stock-bond) market, $\tilde{\gamma}_t = +1$ for all $t \in [t^*, T]$ or $\tilde{\gamma}_t = -1$ for all $t \in [t^*, T]$, $\tilde{\delta}_t = \tilde{\delta}_{t^*}$ for all $t \in [t^*, T]$ is chosen such that $\tilde{\alpha}_{t^*} S_{t^*} + (\tilde{\beta}_{t^*} + \tilde{\delta}_{t^*}) B_{t^*} + \tilde{\gamma}_{t^*} C_{t^*} = 0$. Under such a strategy, the investor does nothing prior to time t^*. On A^c, the investor continues to do nothing for the remaining interval $[t^*, T]$. On A, if $\tilde{\gamma}_{t^*} = +1$, at t^* the investor buys one contingent claim for C_{t^*} and invests $-C_{t^*}$ in the stock-bond market according to $(\tilde{\alpha}_t, \tilde{\beta}_t + \tilde{\delta}_t)$ for $t \in [t^*, T]$. That is, on A, for $t \in [t^*, T]$, the investor follows the strategy $(\tilde{\alpha}, \tilde{\beta})$ in the primary market and, in addition, purchases $\tilde{\delta}_{t^*}$ units of the bond at time t^* and holds those until time T. On A, if $\tilde{\gamma}_{t^*} \equiv -1$, instead of buying one contingent claim at time t^*, the investor sells one contingent claim at t^* and invests the proceeds C_{t^*} according to $(\tilde{\alpha}_t, \tilde{\beta}_t + \tilde{\delta}_t)$ for $t \in [t^*, T]$. Since the value of the strategy ψ is zero at time t^* and $\tilde{\delta}_t$ and $\tilde{\gamma}_t$ do not vary with $t \geq t^*$, the strategy will be naturally self-financing, since $(\tilde{\alpha}, \tilde{\beta})$ is assumed to have this property.

To show existence of the price process, we will prove below that when the price process $C = \{C_t, t \in [0,T]\}$ is a continuous modification of

$$e^{rt} E^{P^*}[X^* \mid \mathcal{F}_t], \quad t \in [0,T], \tag{4.47}$$

it is an arbitrage free price process. For this, we need constraints on the admissible strategies similar to what we had in the case of the primary market to rule out doubling type strategies. Note that in this case, if $\phi^* = (\alpha^*, \beta^*)$ is a replicating strategy for X, then $C_t^* = e^{-rt} C_t = V_t^*(\phi^*)$ for

$t \in [0,T]$ defines a continuous martingale under P^* and P^*-a.s.,

$$dC_t^* = dV_t^*(\phi^*) = \alpha_t^* \, dS_t^* = \sigma \alpha_t^* S_t^* \, d\tilde{W}_t. \tag{4.48}$$

Consequently, in this case, we let the class of admissible strategies Ψ_C be the set of three-dimensional processes $\psi = (\alpha, \beta, \gamma)$ such that

(i) $\psi : [0,T] \times \Omega \to \mathbb{R}^3$ is $(\mathcal{B}_T \times \mathcal{F}_T)$-measurable, where $\psi(t,\omega) = \psi_t(\omega)$ for $t \in [0,T]$ and $\omega \in \Omega$;

(ii) ψ is adapted, i.e., $\psi_t \in \mathcal{F}_t$ for each $t \in [0,T]$;

(iii) $\int_0^T \alpha_t^2 \, dt < \infty$, $\int_0^T |\beta_t| \, dt < \infty$, $\int_0^T \gamma_t^2 (\alpha_t^*)^2 \, dt < \infty$, P^*-a.s.;

(iv) the value process $\{V_t(\psi), t \in [0,T]\}$ defined by

$$V_t(\psi) = \alpha_t S_t + \beta_t B_t + \gamma_t C_t, \quad t \in [0,T], \tag{4.49}$$

is a continuous, adapted process such that the discounted value process $\{V_t^*(\psi), t \in [0,T]\}$ defined by

$$V_t^*(\psi) = V_t(\psi)/B_t, \quad t \in [0,T], \tag{4.50}$$

satisfies P^*-a.s. for each $t \in [0,T]$:

$$\begin{aligned} V_t^*(\psi) &= V_0(\psi) + \int_0^t \alpha_s \, dS_s^* + \int_0^t \gamma_s \, dC_s^*, \\ &= V_0(\psi) + \sigma \int_0^t \alpha_s \, S_s^* \, d\tilde{W}_s + \sigma \int_0^t \gamma_s \, \alpha_s^* \, S_s^* \, d\tilde{W}_s; \end{aligned} \tag{4.51}$$

(v) $\{V_t^*(\psi), \mathcal{F}_t, t \in [0,T]\}$ is a martingale under P^*.

Properties (i)–(iii) above ensure that the integrals appearing in (4.51) are well defined, property (iv) is the self-financing property of ψ (cf. Lemma 4.2.1), and (v) amounts to a constraint on the value of ψ to rule out doubling strategies. Note that since S^* and C^* are already martingales under P^*, this last condition amounts to some control on the size of ψ. One can verify that strategies of the form (4.46) satisfy the above properties (i)–(v) when C is a continuous modification of (4.47).

Given a price process C for the contingent claim, an *arbitrage opportunity* in the stock-bond-contingent claim market is an admissible strategy $\psi \in \Psi_C$ such that $V_0(\psi) = 0$, $V_T(\psi) \geq 0$ and $P^*(V_T(\psi) > 0) > 0$. (Since P and P^* are equivalent, we could just as easily have used P in place of P^* in the last inequality.)

In the following theorem, uniqueness means uniqueness up to indistinguishability of stochastic processes.

Theorem 4.5.1. *The unique arbitrage free price process for the P^*-integrable European contingent claim X is a continuous modification of*

$$\{e^{rt} E^{P^*}[X^* \mid \mathcal{F}_t], \quad t \in [0,T]\}.$$

Proof. Consider a price process $\{C_t, t \in [0,T]\}$ for the European contingent claim. By Theorem 4.4.2, there is a replicating strategy $\phi^* = (\alpha^*, \beta^*)$ for X, and its value process $\{V_t(\phi^*), t \in [0,T]\}$ is a continuous modification of $\{e^{rt} E^{P^*}[X^* | \mathcal{F}_t], t \in [0,T]\}$. Suppose that

$$P^*(C_t \neq V_t(\phi^*) \text{ for some } t \in [0,T]) > 0.$$

Since the process C is assumed to be right continuous and $V(\phi^*)$ is continuous, there exists a $t^* \in [0,T]$ such that $P^*(C_{t^*} \neq V_{t^*}(\phi^*)) > 0$. Let $A = \{\omega \in \Omega : C_{t^*}(\omega) > V_{t^*}(\phi^*)(\omega)\}$ and $\tilde{A} = \{\omega \in \Omega : C_{t^*}(\omega) < V_{t^*}(\phi^*)(\omega)\}$. Then either $P^*(A) > 0$ or $P^*(\tilde{A}) > 0$. First suppose that $P^*(A) > 0$. Define $\psi = \{(\alpha_t, \beta_t, \gamma_t), t \in [0,T]\}$ by

$$\psi_t(\omega) = \begin{cases} (0,0,0), & t < t^*, \\ (0,0,0), & t \geq t^*,\ \omega \in A^c, \\ \left(\alpha_t^*(\omega), \beta_t^*(\omega) + \frac{C_{t^*}(\omega) - V_{t^*}(\phi^*)(\omega)}{B_{t^*}}, -1\right), & t \geq t^*,\ \omega \in A, \end{cases}$$

or in other words, for $t \in [0,T]$, $\omega \in \Omega$,

$$\psi_t(\omega) = 1_{\{t \geq t^*\}} 1_A(\omega) \left(\alpha_t^*, \beta_t^* + \frac{C_{t^*} - V_{t^*}(\phi^*)}{B_{t^*}}, -1\right)(\omega). \tag{4.52}$$

Clearly, the value of the portfolio represented by ψ_t is zero for $t \leq t^*$, and so the value at time zero is zero. Moreover, the value of ψ at time T is zero on A^c. Also, since ϕ^* is a replicating strategy for X, on A, the value at time T is $\left(\frac{C_{t^*} - V_{t^*}(\phi^*)}{B_{t^*}}\right) B_T$, which is strictly positive. Since $P^*(A) > 0$, ψ is an arbitrage opportunity.

Similarly, if $P^*(\tilde{A}) > 0$, then ψ defined for $t \in [0,T]$, $\omega \in \Omega$ by

$$\psi_t(\omega) = 1_{\{t \geq t^*\}} 1_{\tilde{A}}(\omega) \left(-\alpha_t^*, -\beta_t^* + \frac{V_{t^*}(\phi^*) - C_{t^*}}{B_{t^*}}, 1\right)(\omega), \tag{4.53}$$

is an arbitrage opportunity.

Thus, we have shown that the only possible arbitrage free price process (up to indistinguishability) is given by $C_t = V_t(\phi^*)$, $t \in [0,T]$. Next we show that this price process is arbitrage free. For a contradiction, suppose that with this price process, there is $\psi \in \Psi_C$ such that $V_0(\psi) = 0$, $V_T(\psi) \geq 0$ and $E^{P^*}[V_T(\psi)] > 0$. Now, by the admissibility assumption on ψ, the discounted value process $V^*(\psi)$ is a martingale under P^* and so

$$0 = V_0^*(\psi) = E^{P^*}[V_T^*(\psi)] = e^{-rT} E^{P^*}[V_T(\psi)] > 0, \tag{4.54}$$

a contradiction. Thus, there cannot be an arbitrage opportunity with this price process. □

4.6. European Call Option — Black-Scholes Formula

In this section, we apply the results of the previous section to compute the arbitrage free price process for a European call option, and we also identify a replicating strategy for the option.

Let $X = (S_T - K)^+$, which represents the value of a *European call option* with strike price $K \in (0, \infty)$ and expiration time T. Notice that $0 \leq X \leq S_T$ and $E^{P^*}[S_T] = S_0 e^{rT} < \infty$.

4.6.1. Price Process. By Theorem 4.5.1, the arbitrage free price process for the European call option X is given by $\{C_t, t \in [0,T]\}$, which is a continuous adapted process such that for each $t \in [0,T]$, if $C_t^* = C_t e^{-rt}$, then we have P^*-a.s.,

$$C_t^* = E^{P^*}[X^* \mid \mathcal{F}_t] = E^{P^*}\left[(S_T^* - K^*)^+ \mid \mathcal{F}_t\right], \tag{4.55}$$

where $K^* = e^{-rT}K$. It follows from (4.33) that P^*-a.s., for each $t \in [0,T]$,

$$\begin{aligned} S_T^* &= S_t^* \frac{S_0 \exp\left(\sigma \tilde{W}_T - \frac{1}{2}\sigma^2 T\right)}{S_0 \exp\left(\sigma \tilde{W}_t - \frac{1}{2}\sigma^2 t\right)} \\ &= S_t^* \exp\left(\sigma\left(\tilde{W}_T - \tilde{W}_t\right) - \frac{1}{2}\sigma^2 (T-t)\right). \end{aligned}$$

On substituting this into the right member of (4.55), we obtain that for $t \in [0,T]$, P^*-a.s.,

$$C_t^* = E^{P^*}\left[\left(S_t^* \exp\left(\sigma\left(\tilde{W}_T - \tilde{W}_t\right) - \frac{1}{2}\sigma^2(T-t)\right) - K^*\right)^+ \Bigg| \mathcal{F}_t\right]. \tag{4.56}$$

Using the fact that $S_t^* \in \mathcal{F}_t$ and that $\tilde{W}_T - \tilde{W}_t$ is a normal random variable with mean zero and variance $T-t$ that is independent of $\mathcal{F}_t$ under P^* (since $\mathcal{F}_t$ is generated by $\tilde{W}_{\cdot \wedge t}$ and the P^*-null sets), it follows that for $t \in [0,T)$, P^*-a.s.,

$$C_t^* = \frac{1}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} \left(S_t^* e^{\sigma y - \frac{1}{2}\sigma^2(T-t)} - K^*\right)^+ e^{\frac{-y^2}{2(T-t)}} dy,$$

and $C_T^* = (S_T^* - K^*)^+$. Let $z = y/\sqrt{T-t}$. Then the expression on the right above becomes

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(S_t^* e^{\sigma z \sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)} - K^*\right)^+ e^{-\frac{1}{2}z^2} dz. \tag{4.57}$$

Define $f : [0,T] \times (0,\infty) \to [0,\infty)$ by $f(T,x) = (x - K^*)^+$ for all $x \in (0,\infty)$, and for $t \in [0,T)$, $x \in (0,\infty)$,

$$f(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(x e^{\sigma z \sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)} - K^*\right)^+ e^{-\frac{1}{2}z^2} dz. \tag{4.58}$$

Then, by the above, for each $t \in [0, T]$, P^*-a.s.,

$$C_t^* = f(t, S_t^*). \tag{4.59}$$

For $t \in [0, T)$ and $x \in (0, \infty)$, let $l_t(x)$ be such that the integrand in (4.58) is strictly positive if and only if $z > l_t(x)$; i.e.,

$$l_t(x) = \frac{\log\left(\frac{K^*}{x}\right)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{(T-t)}. \tag{4.60}$$

Let Φ denote the cumulative distribution function for a standard normal random variable with mean zero and variance one:

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}u^2}\, du, \quad z \in (-\infty, \infty). \tag{4.61}$$

Then, for $t \in [0, T)$ and $x \in (0, \infty)$,

$$\begin{aligned} f(t,x) &= \frac{x}{\sqrt{2\pi}} \int_{l_t(x)}^{\infty} e^{\sigma z\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t) - \frac{1}{2}z^2}\, dz - \frac{K^*}{\sqrt{2\pi}} \int_{l_t(x)}^{\infty} e^{-\frac{1}{2}z^2}\, dz \\ &= \frac{x}{\sqrt{2\pi}} \int_{l_t(x)}^{\infty} e^{-\frac{1}{2}(z-\sigma\sqrt{T-t})^2}\, dz - K^*\Phi(-l_t(x)) \\ &= \frac{x}{\sqrt{2\pi}} \int_{l_t(x)-\sigma\sqrt{T-t}}^{\infty} e^{-\frac{1}{2}u^2}\, du - K^*\Phi(-l_t(x)) \\ &= x\Phi\left(-l_t(x) + \sigma\sqrt{T-t}\right) - K^*\Phi(-l_t(x)). \end{aligned}$$

Here we have used the change of variable $u = z - \sigma\sqrt{T-t}$ and the symmetry of Φ. Thus, for $t \in [0, T)$, P^*-a.s.,

$$\begin{aligned} C_t^* &= f(t, S_t^*) \\ &= S_t^*\, \Phi\left(\frac{\log\left(\frac{S_t^*}{K^*}\right)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}\right) \\ &\quad - K^*\, \Phi\left(\frac{\log\left(\frac{S_t^*}{K^*}\right)}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t}\right), \end{aligned}$$

and in particular, P^*-a.s.,

$$\begin{aligned} C_0 &= C_0^* \\ &= S_0\, \Phi\left(\frac{\log\left(\frac{S_0}{K^*}\right)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}\right) \\ &\quad - K^*\, \Phi\left(\frac{\log\left(\frac{S_0}{K^*}\right)}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T}\right), \end{aligned} \tag{4.62}$$

which is the famous *Black-Scholes formula*.

4.6.2. Implied Volatility. Note that the Black-Scholes formula (4.62) does not involve the rate of return μ of the stock. The only stock price parameter that appears in the formula is the volatility σ. Given this volatility and assuming the Black-Scholes model for the stock price, one can use the Black-Scholes formula to price a European call option with a given strike price K and expiration time T. For this application, one might hope to use historical data to estimate the volatility parameter. However, such estimates typically produce biased option prices. Another method is often used to estimate volatility; namely, the Black-Scholes formula is used in reverse to obtain an estimate of the volatility of the underlying stock price process from market prices of frequently traded options; e.g., those traded on an exchange. (This is generally a meaningful procedure, as the initial arbitrage free price C_0 given by (4.62) can be verified by differentiation to be a strictly increasing function of the volatility σ for fixed K and T.) An estimate of volatility obtained in this way is called *implied volatility.* This can then be used as an input to the Black-Scholes model to determine prices of other European contingent claims that are not traded frequently.

All European call options based on a given stock use the same volatility parameter σ for their pricing, regardless of the strike price or the expiration date of the option. This would yield a horizontal line for the graph of volatility as a function of strike price. On the other hand, empirical plots of implied volatility as a function of strike price often have a U-shape or half U-shape, referred to as a *smile* or *smirk.* Various efforts have been made to generalize the Black-Scholes model to overcome this deficiency. One approach involves generalizing the model to allow the constant volatility σ to be replaced by a continuous function of a one-dimensional Itô process Z where the driving Brownian motion W' for Z is one-dimensional and W' may be correlated with the Brownian motion W driving the stock price process S. Such models are called stochastic volatility models. They are generally *incomplete,* in that not all suitably integrable contingent claims can be replicated. For a discussion of such models, see Chapter 6 of Musiela and Rutkowski [**32**], and for an in-depth treatment, see the monograph by Fouque, Papanicolaou and Sircar [**17**]. An example of a stochastic volatility model is given in the Exercises for Chapter 5.

4.6.3. Replicating Strategy. The Black-Scholes formula gives the arbitrage free initial price for the European call option, but it does not immediately give a replicating or hedging strategy for the option. Recall that the existence of such a hedging strategy was guaranteed by the martingale representation theorem (cf. the proof of Theorem 4.4.2), but that result is non-constructive. We will now use stochastic calculus and the function f defined above to obtain such a hedging strategy.

Note that $f : [0,T] \times (0,\infty) \to [0,\infty)$ is defined for $t \in [0,T)$, $x \in (0,\infty)$ by

$$\begin{aligned} f(t,x) &= x\Phi\left(\frac{\log\left(\frac{x}{K^*}\right)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}\right) \\ &\quad -K^*\Phi\left(\frac{\log\left(\frac{x}{K^*}\right)}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t}\right) \end{aligned}$$

and

$$f(T,x) = (x-K^*)^+. \tag{4.63}$$

It is straightforward to verify from the above formula that $f \in C^{1,2}([0,T) \times (0,\infty))$; i.e., as a function of $(t,x) \in [0,T) \times (0,\infty)$, $f(t,x)$ is once continuously differentiable in t and twice continuously differentiable in x. In particular, the first partial derivative with respect to x is given for $t \in [0,T)$, $x \in (0,\infty)$, by

$$f_x(t,x) = \Phi\left(\frac{\log\left(\frac{x}{K^*}\right)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}\right), \tag{4.64}$$

which is a bounded continuous function on $[0,T) \times (0,\infty)$. In addition, by using the alternative representation (cf. (4.56)):

$$f(t,x) = E^{P^*}\left[\left(x\exp\left(\sigma(\tilde{W}_T - \tilde{W}_t) - \frac{1}{2}\sigma^2(T-t)\right) - K^*\right)^+\right]$$

for $(t,x) \in [0,T] \times (0,\infty)$, the uniform integrability of $\{\exp(\sigma(\tilde{W}_T - \tilde{W}_t) - \frac{1}{2}\sigma^2(T-t)), t \in [0,T]\}$, and dominated convergence, we can readily verify that f is continuous on $[0,T] \times (0,\infty)$. Note that by the continuity of f, S_t^*, and C_t^*, we may assume that P^*-a.s., $C_t^* = f(t,S_t^*)$ for all $t \in [0,T]$, (i.e., the exceptional P^*-null set does not depend on t). We can apply Itô's formula to $f(t,S_t^*)$ for $t < T$ to obtain P^*-a.s., for all $t < T$,

$$\begin{aligned} f(t,S_t^*) &= f(0,S_0^*) + \int_0^t f_s(s,S_s^*)\,ds + \int_0^t f_x(s,S_s^*)\,dS_s^* \\ &\quad +\frac{1}{2}\int_0^t f_{xx}(s,S_s^*)\,d[S^*]_s, \end{aligned} \tag{4.65}$$

where $f_s(s,x)$ denotes the first partial derivative of $f(s,x)$ with respect to s; $f_x(s,x)$ and $f_{xx}(s,x)$ denote the first and second partial derivatives, respectively, of $f(s,x)$ with respect to x; and $[S^*]$ denotes the quadratic variation process associated with S^* which satisfies $d[S^*]_s = \sigma^2(S_s^*)^2\,ds$ (cf. (4.34)). The first and third integrals above are path-by-path Riemann integrals, and the second integral is a stochastic integral. Indeed, for each $t < T$, $\{\int_0^u f_x(s,S_s^*)\,dS_s^*, \mathcal{F}_u, u \in [0,t]\}$ is a continuous local martingale

starting from zero under P^*. On the other hand, $\{C_u^*, \mathcal{F}_u, u \in [0,t]\}$ is a continuous process and by (4.55) it is a martingale under P^*. Now, P^*-a.s., $C_u^* = f(u, S_u^*)$ for all $u \in [0,t]$, and so by (4.65) we have P^*-a.s. for all $u \in [0,t]$:

$$\begin{aligned}\int_0^u \left(f_s(s, S_s^*) + \frac{1}{2}\sigma^2 (S_s^*)^2 f_{xx}(s, S_s^*) \right) ds \\ = C_u^* - C_0^* - \int_0^u f_x(s, S_s^*)\, dS_s^*. \end{aligned} \tag{4.66}$$

Now, by the above discussion, the right side of the equals sign above defines a continuous local martingale under P^* that starts from zero, and the left side defines a continuous, adapted process whose paths are of bounded variation on $[0,t]$. It follows (cf. [**11**], Corollary 4.5), that P^*-a.s., both sides of (4.66) are zero for all $u \in [0,t]$. Since $t < T$ was arbitrary, this in fact holds true for all $u \in [0,T)$. Indeed, not only is the left side zero P^*-a.s., but one can show (for example by direct verification using the formula for f) that f satisfies the *Black-Scholes partial differential equation:*

$$f_t + \frac{1}{2}\sigma^2 x^2 f_{xx} = 0 \qquad \text{for all } (t,x) \in [0,T) \times (0,\infty).$$

Remark. There are several different forms of the Black-Scholes partial differential equation, depending on whether one uses the function $f(t,x)$ for representing C^* or one uses the undiscounted function $e^{rt} f(t,x)$ for representing C.

Using the fact that P^*-a.s., the right side of (4.66) is zero for all $u \in [0,T)$, we have that P^*-a.s.,

$$C_t^* = C_0 + \int_0^t f_x(s, S_s^*)\, dS_s^* \qquad \text{for all } t < T. \tag{4.67}$$

(Note that $C_0 = C_0^*$ for this.) Comparing this with the self-financing property of trading strategies and keeping in mind that we want C^* to be the discounted value process for a replicating strategy, we define

$$\alpha_t = f_x(t, S_t^*) \qquad \text{for each } t \in [0,T).$$

The definition of α_T is not critical (since it is only in force for an instant of time) as long as it is part of a self-financing strategy. However, we can define it in such a way that $\alpha_T = \lim_{t \to T} \alpha_t$ P^*-a.s. Note that since $P^*(S_T^* = K^*) = 0$ and S^* has continuous paths, we have that P^*-a.s.,

$$\begin{aligned} \lim_{t \to T} f_x(t, S_t^*) &= \lim_{t \to T} \Phi \left(\frac{\log\left(\frac{S_t^*}{K^*}\right)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t} \right) \\ &= 1_{\{S_T^* > K^*\}}. \end{aligned} \tag{4.68}$$

Thus, we may and do define $\alpha_T = 1_{\{S_T^* > K^*\}}$. In order to maintain the appropriate value process, we define

$$\beta_t = C_t^* - \alpha_t S_t^*, \qquad t \in [0, T].$$

It is straightforward to verify that $\phi = \{(\alpha_t, \beta_t), t \in [0, T]\}$ is a trading strategy with discounted value process C^*. By the continuity of C^* and of the stochastic integral process $\{\int_0^t \alpha_s \, dS_s^*, t \in [0, T]\}$, it follows on taking the limit as $t \to T$ in (4.67) that we have P^*-a.s., for each $t \in [0, T]$,

$$V_t^*(\phi) = \alpha_t S_t^* + \beta_t = C_t^* = C_0 + \int_0^t \alpha_s \, dS_s^*,$$

which implies that ϕ is self-financing. Moreover, since $\{C_t^*, \mathcal{F}_t, t \in [0, T]\}$ is a martingale under P^*, ϕ is an admissible strategy. Finally, since $V_T(\phi) = C_T = (S_T - K)^+$, ϕ is a replicating strategy for the European call option with strike price K and expiration time T. The results of this section are summarized in the following theorem.

Theorem 4.6.1. *For each $t \in [0, T)$, let*

$$\begin{aligned} C_t^* &= S_t^* \, \Phi\left(\frac{\log\left(\frac{S_t^*}{K^*}\right)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t} \right) \\ &\quad - K^* \, \Phi\left(\frac{\log\left(\frac{S_t^*}{K^*}\right)}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t} \right), \end{aligned} \tag{4.69}$$

and let $C_T^* = (S_T^* - K^*)^+$. Define $\alpha_T = 1_{\{S_T > K\}}$, and let

$$\begin{aligned} \alpha_t &= \Phi\left(\frac{\log\left(\frac{S_t^*}{K^*}\right)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t} \right), \quad \text{for each } t \in [0, T), \\ \beta_t &= C_t^* - \alpha_t S_t^* \quad \text{for each } t \in [0, T]. \end{aligned}$$

Then $\phi = \{(\alpha_t, \beta_t), t \in [0, T]\}$ is a replicating strategy for the European call option with strike price K and expiration time T, and the arbitrage free price process for this option is given by $\{e^{rt} C_t^*, t \in [0, T]\}$.

4.7. American Contingent Claims

An *American contingent claim* (ACC) is represented by a non-negative, continuous, adapted stochastic process $Y = \{Y_t, t \in [0, T]\}$ evolving in continuous time such that $E^{P^*}\left[\sup_{0 \le t \le T} Y_t\right] < \infty$. The random variable Y_t, $t \in [0, T]$, is interpreted as the payoff for the claim if the owner cashes it in at time t. The time at which the owner cashes in the claim is required to be a stopping time taking values in $[0, T]$; i.e., it is a random variable

$\tau : \Omega \to [0, T]$ such that $\{\tau \leq t\} \in \mathcal{F}_t$ for each $t \in [0, T]$. For $0 \leq s \leq t \leq T$, let $\mathcal{T}_{[s,t]}$ denote the set of stopping times that take values in the interval $[s, t]$. An example of an American contingent claim is an *American call option* with strike price $K \in (0, \infty)$, which has payoff $Y_t = (S_t - K)^+$ at time t, $t \in [0, T]$. Note that if $S_t \leq K$, cashing in the contingent claim at time t is equivalent to not exercising the option at all. We have adopted this convention so that we can use one framework for treating all contingent claims, including options and contracts.

As noted in Section 2.3, devoted to pricing American contingent claims in the setting of the discrete binomial model, an important feature of an American contingent claim is that the buyer and the seller of such a derivative have different actions available to them: the buyer may cash in the claim at any stopping time $\tau \in \mathcal{T}_{[0,T]}$, whereas the seller seeks protection from the risk associated with all possible choices of the stopping time τ by the buyer. Also, as for the binomial model, in pricing and hedging American contingent claims, a critical role is played by a superhedging strategy that hedges the risk for the seller of an American contingent claim.

A *superhedging strategy* for the seller of an American contingent claim, $Y = \{Y_t, t \in [0, T]\}$, is an admissible strategy $\phi = \{(\alpha_t, \beta_t), t \in [0, T]\}$ with value $V_t(\phi)$ at time $t \in [0, T]$, such that $V_t(\phi) \geq Y_t$ for each $t \in [0, T]$, P^*-a.s. In the context of the binomial model, such a ϕ can be constructed stepwise by proceeding backwards through the binary tree. In continuous time, such a stepwise construction is no longer available. Nevertheless, one can exploit the theory of Snell envelopes to show that a superhedging strategy exists.

For a non-empty collection of non-negative random variables $\{X_\gamma, \gamma \in \Gamma\}$ defined on a probability space, their *essential supremum* is a random variable X satisfying

(i) $X \geq X_\gamma$ a.s. for each $\gamma \in \Gamma$, and

(ii) if Z is a random variable such that $Z \geq X_\gamma$ a.s. for each $\gamma \in \Gamma$, then $Z \geq X$ a.s.

We write $X = \text{ess sup}_{\gamma \in \Gamma} X_\gamma$. (For the existence and properties of the essential supremum, see Karatzas and Shreve [**28**], Appendix A. The essential supremum is unique, up to a.s. equivalence.)

For each $t \in [0, T]$, let $Y_t^* = \frac{Y_t}{B_t} = e^{-rt} Y_t$. Define a process $U^* = \{U_t^*, t \in [0, T]\}$ as follows. For each $t \in [0, T]$, let

$$U_t^* = \text{ess sup}_{\tau \in \mathcal{T}_{[t,T]}} E^{P^*} \left[Y_\tau^* \mid \mathcal{F}_t \right], \tag{4.70}$$

where we take U_t^* to be $\mathcal{F}_t$-measurable, without loss of generality. Also, for $t = 0$, since $\mathcal{F}_0$ is trivial, we may take $U_0^* = \sup_{\tau \in \mathcal{T}_{[0,T]}} E^{P^*}[Y_\tau^*]$, and for $t = T$, we may take $U_T^* = Y_T^*$. The right member in (4.70) is finite P^*-a.s.

since we assumed that $Y_s \geq 0$ for each $s \in [0,T]$ and $E^{P^*}[\sup_{0\leq s\leq T} Y_s] < \infty$. It is straightforward to verify that $\{U_t^*, \mathcal{F}_t, t \in [0,T]\}$ is a supermartingale. In fact (cf. [**28**], Appendix D), there is a modification of this supermartingale whose paths are right continuous with finite left limits (abbreviated as r.c.l.l.). Henceforth, we shall assume that U^* is such a "nice" modification. Let $\tau^*(t) \equiv \inf\{s \geq t : U_s^* = Y_s^*\}$ for each $t \in [0,T]$ and define $\tau^* = \tau^*(0)$. Note that $U_{\tau^*}^* = Y_{\tau^*}^*$ P^*-a.s., since U^* has right continuous paths, Y^* has continuous paths, and $U_T^* = Y_T^*$ P^*-a.s. For later use, we define

$$U_t = e^{rt} U_t^*, \quad t \in [0,T]. \tag{4.71}$$

The following result is proved in Appendix D of Karatzas and Shreve [**28**].

Lemma 4.7.1.

(i) $\{U_t^*, \mathcal{F}_t, t \in [0,T]\}$ *is the smallest* P^**-supermartingale that has r.c.l.l. paths and satisfies* P^**-a.s.* $U_t^* \geq Y_t^*$ *for each* $t \in [0,T]$*;*

(ii) $\tau^*(t) \in \mathcal{T}_{[t,T]}$ *and*

$$U_t^* = E^{P^*}\left[Y_{\tau^*(t)}^* \mid \mathcal{F}_t\right], \qquad t \in [0,T];$$

(iii) $\{U_{t\wedge\tau^*}^*, \mathcal{F}_t, t \in [0,T]\}$ *is a martingale under* P^*.

By the Doob-Meyer decomposition theorem (cf. [**28**], Theorem D.13), we have the following decomposition for the uniformly integrable supermartingale U^*:

$$U_t^* = U_0^* + M_t - A_t, \qquad t \in [0,T], \tag{4.72}$$

where $M = \{M_t, t \in [0,T]\}$ is a P^*-martingale with r.c.l.l. paths; $M_0 = 0$; and $A = \{A_t, t \in [0,T]\}$ is a continuous, non-decreasing, adapted process such that $A_0 = 0$, $E^{P^*}[A_t] < \infty$ for each $t \in [0,T]$. In fact, this decomposition is unique (up to indistinguishability of stochastic processes). By the martingale representation theorem (Theorem 4.4.1), there is a $(\mathcal{B}_T \times \mathcal{F}_T)$-measurable, adapted process $\eta = \{\eta_t, t \in [0,T]\}$ such that P^*-a.s., $\int_0^T \eta_s^2 ds < \infty$ and

$$M_t = \int_0^t \eta_s \, d\tilde{W}_s, \qquad \text{for all } t \in [0,T]. \tag{4.73}$$

In particular, P^*-a.s., the paths of M are continuous, and so without loss of generality we may assume that all of the paths of M are continuous.

Let

$$\phi_t = (\alpha_t, \beta_t) = \left(\frac{\eta_t}{\sigma S_t^*}, U_0^* + M_t - \frac{\eta_t}{\sigma}\right), \qquad t \in [0,T]. \tag{4.74}$$

Then, ϕ is a trading strategy since it satisfies the measurability and local integrability properties required of a trading strategy (cf. Section 4.2). The

discounted value process for ϕ satisfies P^*-a.s. for each $t \in [0,T]$:

$$\begin{aligned} V_t^*(\phi) &= \frac{\eta_t}{\sigma} + U_0^* + M_t - \frac{\eta_t}{\sigma} \\ &= U_0^* + M_t \\ &= U_0^* + \int_0^t \eta_s \, d\tilde{W}_s \\ &= U_0^* + \int_0^t \alpha_s \sigma S_s^* \, d\tilde{W}_s \\ &= V_0(\phi) + \int_0^t \alpha_s \, dS_s^*, \end{aligned} \tag{4.75}$$

where the third line is by (4.73), the fourth is by (4.74), and the fifth is by (4.34). We also used the fact that $U_0^* = V_0^*(\phi) = V_0(\phi)$. It follows from Lemma 4.2.1 that ϕ is self-financing. Moreover, since U_0^* is a finite constant and M is a martingale under P^*, $\{V_t^*(\phi), \mathcal{F}_t, t \in [0,T]\}$ is a P^*-martingale. Hence, ϕ is an admissible strategy.

Next we verify that ϕ is a superhedging strategy. For this note that by (4.75), (4.72) and (i) of Lemma 4.7.1, we have P^*-a.s.,

$$V_t^*(\phi) = U_0^* + M_t = U_t^* + A_t \geq U_t^* \geq Y_t^*, \qquad t \in [0,T]. \tag{4.76}$$

Finally, by Lemma 4.7.1, (4.72), and the uniqueness of the Doob-Meyer decomposition, P^*-a.s.,

$$U_{t\wedge\tau^*}^* = U_0^* + M_{t\wedge\tau^*} = V_{t\wedge\tau^*}^*(\phi), \quad t \in [0,T]. \tag{4.77}$$

We now argue that $U_0 = U_0^*$ is the unique arbitrage free initial price for the American contingent claim at time zero. For this, we need the notion of an arbitrage in a market where the stock and bond can be traded, and the American contingent claim (ACC) can be bought or sold at time zero. For such a market, let the initial price of the ACC be a constant C_0. There are two types of arbitrage opportunities: one for a seller and another for a buyer of the ACC. The *seller* of the ACC has an *arbitrage opportunity* if there is an admissible strategy ϕ^s in stock and bond such that $V_0(\phi^s) = C_0$ and for all stopping times $\tau \in \mathcal{T}_{[0,T]}$,

$$V_\tau(\phi^s) - Y_\tau \geq 0 \qquad \text{and} \qquad E^{P^*}\left[V_\tau(\phi^s) - Y_\tau\right] > 0. \tag{4.78}$$

The *buyer* of the ACC has an *arbitrage opportunity* if there is an admissible strategy ϕ^b such that $V_0(\phi^b) = -C_0$ and there exists a stopping time $\tau \in \mathcal{T}_{[0,T]}$ such that

$$V_\tau(\phi^b) + Y_\tau \geq 0 \qquad \text{and} \qquad E^{P^*}\left[V_\tau(\phi^b) + Y_\tau\right] > 0. \tag{4.79}$$

In conditions (4.78)–(4.79), one could equivalently evaluate the expectations under P instead of P^*, since P and P^* have the same sets of probability

zero. The initial price C_0 is arbitrage free if there is no arbitrage opportunity for a seller or buyer of the contingent claim at this initial price.

To take advantage of a seller's arbitrage opportunity, an investor could sell one ACC at time zero for C_0 and invest the proceeds C_0 according to the trading strategy ϕ^s until the claim is cashed in by the buyer at some stopping time τ. At time τ, the seller would give the buyer Y_τ to pay off the claim and could put the remainder of his wealth $V_\tau(\phi^s) - Y_\tau$ in bond for the time period $(\tau, T]$. This would result in a final value of $(V_\tau(\phi^s) - Y_\tau)e^{r(T-\tau)}$, which is non-negative and is strictly positive with positive probability under P^*.

To take advantage of a buyer's arbitrage opportunity, an investor could buy one ACC at time zero for C_0 and invest $-C_0$ according to ϕ^b until the time τ when the buyer cashes in the claim. The buyer would then have $V_\tau(\phi^b) + Y_\tau$ at time τ and could put this in bond for the time period $(\tau, T]$ so that the buyer's final wealth would be $(V_\tau(\phi^b) + Y_\tau)e^{r(T-\tau)}$, which is non-negative and is strictly positive with positive probability under P^*.

Theorem 4.7.2. *The unique arbitrage free price at time zero for the American contingent claim is U_0, which is defined by*

$$U_0 = \sup_{\tau \in \mathcal{T}_{[0,T]}} E^{P^*}[e^{-r\tau} Y_\tau]. \tag{4.80}$$

Proof. We first show that the arbitrage free price cannot be anything other than U_0; i.e., we establish uniqueness of an arbitrage free initial price. Let ϕ^* be the superhedging strategy defined in (4.74).

Suppose that $C_0 > U_0$. Then there is an arbitrage opportunity for the seller of the American contingent claim. To see this, let ϕ^s denote the self-financing trading strategy in stock and bond corresponding to investing U_0 according to the superhedging strategy ϕ^* and $C_0 - U_0$ in the bond for all time. Note that the initial value $V_0(\phi^s) = C_0$ and, by (4.76), the value $V_t(\phi^*)$ of ϕ^* at time t is at least as great as Y_t for each $t \in [0, T]$. Then for any stopping time $\tau \in \mathcal{T}_{[0,T]}$,

$$\begin{aligned} V_\tau(\phi^s) - Y_\tau &= V_\tau(\phi^*) + (C_0 - U_0)B_\tau - Y_\tau \\ &\geq Y_\tau + (C_0 - U_0)B_\tau - Y_\tau \\ &\geq (C_0 - U_0)B_\tau, \end{aligned}$$

where $(C_0 - U_0)B_\tau > 0$. Thus, there is an arbitrage opportunity for a seller of the American contingent claim.

On the other hand, suppose that $C_0 < U_0$. Then there is an arbitrage opportunity for the buyer of the American contingent claim. To see this, let ϕ^b denote the self-financing trading strategy corresponding to investing $-U_0$ according to the negative $-\phi^*$ of the superhedging strategy ϕ^* and investing

$U_0 - C_0$ in the bond for all time. Furthermore, consider the stopping time τ^* (viewed as the time at which an ACC, bought initially by the buyer, should be cashed in). Note that $V_t(-\phi^*) = -V_t(\phi^*)$ for $t \in [0,T]$, and $V_0(\phi^b) = -C_0$. Then, using the fact that $V_{\tau^*}(\phi^*) = U_{\tau^*} = Y_{\tau^*}$ (cf. (4.77)), we have

$$V_{\tau^*}(\phi^b) + Y_{\tau^*} = -V_{\tau^*}(\phi^*) + (U_0 - C_0)B_{\tau^*} + Y_{\tau^*} = (U_0 - C_0)B_{\tau^*}.$$

Since $(U_0 - C_0)B_{\tau^*} > 0$, there is an arbitrage opportunity for a buyer of the American contingent claim.

Finally, suppose that $C_0 = U_0$. We need to show that $C_0 = U_0$ is arbitrage free; i.e., that there exists an arbitrage free initial price. We begin by showing that there is no arbitrage opportunity for a seller of an American contingent claim with $C_0 = U_0$. For a proof by contradiction, suppose that there exists an admissible strategy ϕ^s such that $V_0(\phi^s) = U_0$ and for each $\tau \in \mathcal{T}_{[0,T]}$, (4.78) holds. Note that $\tau^* \in \mathcal{T}_{[0,T]}$. Then it follows from (4.78), with $\tau = \tau^*$, that $V_{\tau^*}(\phi^s) - Y_{\tau^*} \geq 0$ and strict inequality holds with positive probability under P^*. Multiplying through by $e^{-r\tau^*}$ and taking expectations, we see that

$$E^{P^*}\left[V^*_{\tau^*}(\phi^s) - Y^*_{\tau^*}\right] > 0. \tag{4.81}$$

On the other hand, by Doob's stopping theorem for continuous martingales (cf. Appendix C),

$$E^{P^*}\left[V^*_{\tau^*}(\phi^s)\right] = V^*_0(\phi^s) = U^*_0,$$

and by (ii) in Lemma 4.7.1, $E^{P^*}\left[Y^*_{\tau^*}\right] = U^*_0$. Combining these two properties, we obtain

$$E^{P^*}\left[V^*_{\tau^*}(\phi^s) - Y^*_{\tau^*}\right] = 0,$$

which contradicts (4.81). Therefore, no such ϕ^s exists, and consequently there is no arbitrage opportunity for a seller of the American contingent claim.

Next we show that there is no arbitrage opportunity for a buyer of the American contingent claim with $C_0 = U_0$. For a proof by contradiction, suppose that there exists an admissible strategy ϕ^b such that $V_0(\phi^b) = -U_0$ and a stopping time $\tau \in \mathcal{T}_{[0,T]}$ such that (4.79) holds. Then,

$$E^{P^*}\left[V^*_\tau(\phi^b) + Y^*_\tau\right] > 0.$$

However, by Doob's stopping theorem, $E^{P^*}\left[V^*_\tau(\phi^b)\right] = V^*_0(\phi^b) = -U^*_0$, and, by the definition of U^*_0, we have $E^{P^*}\left[Y^*_\tau\right] \leq U^*_0$. Therefore,

$$E^{P^*}\left[V^*_\tau(\phi^b) + Y^*_\tau\right] \leq 0,$$

which yields the desired contradiction. Hence there is no arbitrage opportunity for a buyer of the American contingent claim. $\square$

4.8. American Call Option

Consider an American call option with payoff process Y given by

$$Y_t = (S_t - K)^+, \qquad t \in [0,T],$$

where K is a fixed constant in $(0,\infty)$. Note that by the Cauchy-Schwarz inequality and Doob's L^2-martingale inequality,

$$\begin{aligned}
E^{P^*}\left[\sup_{0\le t\le T} Y_t\right] &\le e^{rT}E^{P^*}\left[\sup_{0\le t\le T} S_t^*\right] \\
&\le e^{rT}\left(E^{P^*}\left[\sup_{0\le t\le T}(S_t^*)^2\right]\right)^{\frac{1}{2}} \\
&\le 2e^{rT}\left(E^{P^*}\left[(S_T^*)^2\right]\right)^{\frac{1}{2}}, \\
&= 2S_0^2 e^{rT+\frac{1}{2}\sigma^2 T}\left(E^{P^*}\left[\exp\left(2\sigma\tilde{W}_T - \frac{1}{2}(4\sigma^2 T)\right)\right]\right)^{\frac{1}{2}} \\
&= 2S_0^2 e^{rT+\frac{1}{2}\sigma^2 T}.
\end{aligned}$$

For the next to last inequality above we have used (4.33). Thus, Y is a non-negative, continuous, adapted process satisfying $E^{P^*}[\sup_{0\le t\le T} Y_t] < \infty$, as required in the hypotheses of the previous section. Hence, by Theorem 4.7.2, the initial arbitrage free price is given by

$$\begin{aligned}
U_0 &= \sup\nolimits_{\tau\in\mathcal{T}_{[0,T]}} E^{P^*}\left[Y_\tau^*\right], &(4.82)\\
&= \sup\nolimits_{\tau\in\mathcal{T}_{[0,T]}} E^{P^*}\left[e^{-r\tau}(S_\tau - K)^+\right]. &(4.83)
\end{aligned}$$

By part (ii) of Lemma 4.7.1, the supremum in the above expression is achieved by $\tau^* = \tau^*(0) = \inf\{t \in [0,T] : U_t^* = Y_t^*\}$. In general, neither U_0 nor τ^* is easy to compute. However, in this section, we will exploit the convexity and increasing property of the function $g(x) = (x-K)^+$, $x \ge 0$, to show that the initial arbitrage free price for an American call option is the same as that for the European call option with the same strike price and expiration time. One can also show that a buyer of the American call option who does not cash it in until the expiration time T will not create an arbitrage opportunity for the seller of the option. We leave this as an exercise for the reader.

We begin by deriving a stochastic integral equation for the discounted payoff process Y^*. For this, we use a couple of results from stochastic calculus that are beyond the summary provided in Appendix D. We provide references where these results are cited.

Under P^*, $\{S_t = e^{rt}S_t^*, t \geq 0\}$ is an Itô process satisfying the following equation P^*-a.s.:

$$S_t = S_0 + \int_0^t rS_s\,ds + \sigma \int_0^t S_s\,d\tilde{W}_s, \quad t \in [0,T]. \tag{4.84}$$

By the Itô-Tanaka formula (cf. Theorems VI.1.2 and VI.1.5 of Revuz-Yor [**35**]), P^*-a.s.,

$$Y_t = g(S_t) = g(S_0) + \int_0^t 1_{\{S_s > K\}}\,dS_s + \frac{1}{2}L_t, \qquad t \in [0,T], \tag{4.85}$$

where $L = \{L_t, t \in [0,T]\}$ is the local time at K of the stock price process S. This local time is a continuous, non-decreasing, adapted process that measures the amount of time (in units of the quadratic variation of S) that S spends near K. Indeed, P^*-a.s., for each $t \in [0,T]$,

$$L_t = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^t 1_{[K,K+\epsilon)}(S_s)\,d[S]_s, \tag{4.86}$$

where $d[S]_s = \sigma^2 S_s^2 ds$; cf. Revuz and Yor [**35**], Corollary VI.1.9. (The Itô-Tanaka formula is a generalization of Itô's formula from twice continuously differentiable functions to convex functions. It can be derived by passing to a limit in Itô's formula applied to a sequence of smooth functions $\{g_n\}$ that approximate g in a suitable manner. The term in Itô's formula containing the first derivative of such a function converges to the second term after the second equality in (4.85). The term in Ito's formula containing the second derivative of such a function converges to the local time term in (4.85). See Chapter VI of [**35**] for details.)

Since $Y_t^* = e^{-rt}g(S_t)$, $t \in [0,T]$ is the product of a continuous, non-decreasing, adapted process and an Itô process, it obeys the usual product rule for differentiation. (For a justification of this rule of stochastic calculus,

see [**11**], Chapter 5.) Consequently we have P^*-a.s.,

$$\begin{aligned}
Y_t^* &= Y_0^* - \int_0^t re^{-rs} g(S_s)\,ds \\
&\quad + \int_0^t e^{-rs}\left(1_{\{S_s>K\}}\,dS_s + \frac{1}{2}\,dL_s\right) \\
&= Y_0^* - \int_0^t re^{-rs} g(S_s)\,ds \\
&\quad + \int_0^t e^{-rs}\left(1_{\{S_s>K\}}(rS_s\,ds + \sigma S_s\,d\tilde{W}_s) + \frac{1}{2}\,dL_s\right) \\
&= Y_0^* + \int_0^t e^{-rs} 1_{\{S_s>K\}} \sigma S_s\,d\tilde{W}_s \\
&\quad + \int_0^t e^{-rs}\left(1_{\{S_s>K\}} rS_s - rg(S_s)\right) ds + \frac{1}{2}\int_0^t e^{-rs}\,dL_s.
\end{aligned}$$

Here the integral with respect to L is defined pathwise as a continuous, adapted, non-decreasing process. Notice that by the definition of g, we have $1_{\{S_s>K\}} rS_s - rg(S_s) = 1_{\{S_s>K\}} rK$ for all $s \in [0,T]$. Thus, P^*-a.s., for $t \in [0,T]$,

$$\begin{aligned}
Y_t^* &= Y_0^* + \sigma \int_0^t e^{-rs} 1_{\{S_s>K\}} S_s\,d\tilde{W}_s \\
&\quad + rK \int_0^t e^{-rs} 1_{\{S_s>K\}}\,ds + \frac{1}{2}\int_0^t e^{-rs}\,dL_s.
\end{aligned}$$

The first integral above defines a continuous local martingale that starts from zero and the final two terms define continuous, adapted, non-decreasing processes. Moreover, Y_t^* is dominated by the P^*-integrable random variable $\sup_{t\in[0,T]} Y_t$ for each $t \in [0,T]$. It follows that $Y^* = \{Y_t^*, \mathcal{F}_t, t \in [0,T]\}$ is a continuous submartingale. Therefore, by Doob's stopping theorem (cf. Appendix C), for each $\tau \in \mathcal{T}_{[0,T]}$,

$$E^{P^*}[Y_\tau^*] \le E^{P^*}[Y_T^*].$$

This together with (4.82) gives

$$U_0 = \sup\nolimits_{\tau\in\mathcal{T}_{[0,T]}} E^{P^*}[Y_\tau^*] \le E^{P^*}[Y_T^*].$$

But notice that $\tau \equiv T$ is an element of $\mathcal{T}_{[0,T]}$ and so by (4.82),

$$U_0 \ge E^{P^*}[Y_T^*].$$

It follows that the initial arbitrage free price for the American call option with strike price K and expiration time T is given by

$$U_0 = E^{P^*}[Y_T^*] = e^{-rT} E^{P^*}\left[(S_T - K)^+\right],$$

which is the same as the initial arbitrage free price of the European call option with the same strike price and expiration time.

4.9. American Put Option

Consider an American put option with strike price $K \in (0, \infty)$ and expiration time T. Then this has payoff process $Y = \{Y_t, t \in [0, T]\}$, where

$$Y_t = (K - S_t)^+, \qquad t \in [0, T].$$

Since Y is a continuous, adapted process bounded by K, and $Y_t \geq 0$ for all $t \in [0, T]$, Y satisfies the hypotheses required of an American contingent claim in Section 4.7. Thus, by Theorem 4.7.2, the initial arbitrage free price for this American put option is given by

$$U_0 = \sup_{\tau \in \mathcal{T}_{[0,T]}} E^{P^*}[e^{-r\tau}(K - S_\tau)^+]. \tag{4.87}$$

The right member above is difficult to compute directly, as it involves taking a supremum over an uncountable family of stopping times. In fact, for each $t \in [0, T]$, the value of $U_t = e^{rt}U_t^*$, for U_t^* as defined in (4.70), can be characterized in terms of a solution of a free boundary problem for a parabolic partial differential equation. Here we shall motivate and state results associated with this characterization. However, for a detailed development including proofs, we refer the reader to the more advanced text on mathematical finance by Karatzas and Shreve [**28**], Section 2.7. Their treatment is largely based on the original paper of Jacka [**24**].

Recall from (4.33) and (4.10) that under P^*,

$$S_t = S_0 \exp\left(\sigma \tilde{W}_t - \frac{1}{2}\sigma^2 t + rt\right), \quad t \in [0, T], \tag{4.88}$$

where $\tilde{W}$ is a standard Brownian motion. Now, for each $T \in [0, \infty)$ and $x \in [0, \infty)$, let $v(T, x)$ denote the value of the right member of (4.87) given that $S_0 = x$. Note that $v(0, x) = (K - x)^+$. (Although $T > 0$ has so far been fixed in this chapter and $S_0 > 0$, for the definition of $v(T, x)$ we allow T to vary in $[0, \infty)$ and allow $S_0 = 0$ as a possibility.)

Proposition 4.9.1. *The function $v : [0, \infty) \times [0, \infty) \to [0, \infty)$ is continuous and satisfies $v(t, x) \geq (K - x)^+$ for all $t \in [0, \infty)$ and $x \in [0, \infty)$.*

Proof. See Proposition 7.1 in Section 2.7 of [**28**]. □

We now revert to $T \in (0, \infty)$ and $S_0 > 0$ being fixed. Then,

$$U_0 = v(T, S_0). \tag{4.89}$$

In fact, one can express the entire process $\{U_t, t \in [0, T]\}$, which turns out to be the arbitrage free price process for the American put (see the

Exercises), in terms of the function v as follows. For this, we note that the filtration $\{\mathcal{F}_t, t \in [0,T]\}$ is the same as that generated by the process $\{S_t, t \in [0,T]\}$ under P^* (this includes augmentation of the filtration by the P^*-null sets). By the stationary independent increments property of $\tilde{W}$ under P^*, $\{S_t, t \in [0,T]\}$ is a strong Markov process. In particular, using this property, it can be shown that P^*-a.s., for each $t \in [0,T]$,

$$\begin{aligned} U_t &= \text{ess sup}_{\tau \in \mathcal{T}_{[t,T]}} E^{P^*}\left[e^{-r(\tau - t)}(K - S_\tau)^+ \mid \mathcal{F}_t\right] \\ &= v(T-t, S_t). \end{aligned}$$

Recall the definition of τ^* from Section 4.7 and the fact that $\{U^*_{t\wedge\tau^*}, \mathcal{F}_t, t \in [0,T]\}$ is a martingale under P^*. Using the above we have

$$\tau^* = \inf\{s \geq 0 : v(T-s, S_s) = (K - S_s)^+\}. \tag{4.90}$$

This motivates introducing the continuation region

$$\mathcal{C} = \{(t,x) \in (0,\infty) \times (0,\infty) : v(t,x) > (K-x)^+\}$$

and the stopping region

$$\mathcal{D} = \{(t,x) \in [0,\infty) \times [0,\infty) : v(t,x) = (K-x)^+\}.$$

For each $t \in (0,\infty)$, define the section of $\mathcal{C}$ by

$$\mathcal{C}_t = \{x \in (0,\infty) : v(t,x) > (K-x)^+\}. \tag{4.91}$$

Proposition 4.9.2. *For each $t \in (0,\infty)$, there is a number $c(t) \in (0,K)$ such that $\mathcal{C}_t = (c(t),\infty)$. The function $c : (0,\infty) \to (0,K)$ is non-increasing, continuous, and $\lim_{t\downarrow 0} c(t) = K$. We denote this last quantity by $c(0)$.*

Proof. See Section 2.7 of [**28**]. □

The function c defines the "free boundary". It turns out that v satisfies a parabolic partial differential equation on $\mathcal{C}$. In fact, $v(s,x)$ is once continuously differentiable in s and and twice continuously differentiable in x for $(s,x) \in \mathcal{C}$. For $(s,x) \in \mathcal{C}$, we let $v_s(s,x)$ denote the first partial derivative with respect to s and $v_x(s,x)$, $v_{xx}(s,x)$ denote the first, second partial derivatives with respect to x, respectively. Assuming that these partial derivatives exist and are continuous in $\mathcal{C}$, we now motivate the form of the partial differential equation satisfied by v.

For each $\epsilon > 0$, let

$$\tau^*_\epsilon = \inf\left\{s \geq 0 : v(T-s, S_s) \leq (K-S_s)^+ + \epsilon\right\}.$$

Since $v(0,x) = (K-x)^+$, and v and S are continuous, we have $\tau^*_\epsilon < T$. We can apply Itô's formula to the Itô process $\{(T - (t \wedge \tau^*_\epsilon), S_{t\wedge\tau^*_\epsilon}), t \in [0,T]\}$ and the function v defined on $\mathcal{C}$, noting that on $\{\tau^*_\epsilon > 0\}$, the Itô process has paths in $\mathcal{C}$ and that for Itô's formula we need only continuous first partial

derivatives with respect to the first variable in v as the first coordinate of the Itô process has paths of bounded variation (cf. [**11**], Theorem 5.10). Making use of (4.84), we obtain for each $\epsilon > 0$, P^*-a.s., for each $t \in [0,T]$,

$$\begin{aligned} U_{t\wedge\tau_\epsilon^*} &= v(T-(t\wedge\tau_\epsilon^*), S_{t\wedge\tau_\epsilon^*}) \\ &= v(T,S_0) - \int_0^{t\wedge\tau_\epsilon^*} v_s(T-u,S_u)\,du \\ &\quad + \int_0^{t\wedge\tau_\epsilon^*} v_x(T-u,S_u)\,dS_u \\ &\quad + \frac{1}{2}\int_0^{t\wedge\tau_\epsilon^*} v_{xx}(T-u,S_u)\,\sigma^2 S_u^2\,du \\ &= v(T,S_0) + \int_0^{t\wedge\tau_\epsilon^*} v_x(T-u,S_u)\,\sigma S_u\,d\tilde{W}_u \\ &\quad + \int_0^{t\wedge\tau_\epsilon^*} (\mathcal{L}v)(T-u,S_u)\,du, \end{aligned}$$

where for $(s,x) \in \mathcal{C}$,

$$(\mathcal{L}v)(s,x) = -v_s(s,x) + rxv_x(s,x) + \frac{1}{2}\sigma^2 x^2 v_{xx}(s,x).$$

Hence, applying Itô's formula again, we have P^*-a.s., for each $t \in [0,T]$,

$$\begin{aligned} U^*_{t\wedge\tau_\epsilon^*} &= e^{-r(t\wedge\tau_\epsilon^*)} U_{t\wedge\tau_\epsilon^*} \\ &= U_0^* + \int_0^{t\wedge\tau_\epsilon^*} e^{-ru} v_x(T-u,S_u)\,\sigma S_u\,d\tilde{W}_u \\ &\quad + \int_0^{t\wedge\tau_\epsilon^*} e^{-ru}(\tilde{\mathcal{L}}v)(T-u,S_u)\,du, \end{aligned}$$

where for $(s,x) \in \mathcal{C}$,

$$(\tilde{\mathcal{L}}v)(s,x) = (\mathcal{L}v)(s,x) - rv(s,x). \tag{4.92}$$

Now, since $\tau_\epsilon^* \le \tau^*$, by part (iii) of Lemma 4.7.1 and Doob's stopping theorem, we have that $\{U^*_{t\wedge\tau_\epsilon^*}, \mathcal{F}_t, t \in [0,T]\}$ is a right continuous martingale under P^*. Hence it follows by the uniqueness of its Doob-Meyer decomposition (cf. [**27**], Section 1.4) that P^*-a.s., for all $t \in [0,T]$:

$$\int_0^{t\wedge\tau_\epsilon^*} e^{-ru}(\tilde{\mathcal{L}}v)(T-u,S_u)\,du = 0.$$

One way in which this could be satisfied is if $\tilde{\mathcal{L}}v = 0$ on $\mathcal{C}$, since $(T-u, S_u) \in \mathcal{C}$ whenever $0 < u < \tau_\epsilon^*$. In fact, the following free boundary problem characterizes v, together with the boundary function c.

Proposition 4.9.3. *The pair $(f,d) = (v,c)$ is the unique pair of functions satisfying the following:*

(i) $f : [0,\infty)^2 \to [0,\infty)$ *is continuous and such that* $f(t,x)$ *is once continuously differentiable in* t *and twice continuously differentiable in* x *for all* $(t,x) \in \mathcal{G} = \{(t,x) \in (0,\infty) \times (0,\infty) : x > d(t)\}$*;*

(ii) $d : [0,\infty) \to (0,K]$ *is non-increasing and left continuous,* $d(0) = \lim_{x\downarrow 0} d(x) = K$*;*

(iii)

$$\begin{aligned}
\tilde{\mathcal{L}} f &= 0 \quad \text{in } \mathcal{G}, \\
f(t,x) &\geq (K-x)^+, \quad (t,x) \in [0,\infty)^2, \\
f(t,x) &= K - x, \quad 0 \leq t < \infty,\ 0 \leq x \leq d(t), \\
f(0,x) &= (K-x)^+ \quad \text{for } d(0) \leq x < \infty, \\
\lim_{x\to\infty} \max_{0\leq s\leq t} |f(s,x)| &= 0, \quad t \in (0,\infty) \\
\lim_{x\downarrow d(t)} f_x(t,x) &= -1, \quad t \in (0,\infty).
\end{aligned}$$

Proof. See Theorem 7.12 of Section 2.7 in [**28**]. □

While the above proposition provides a characterization of v and the free boundary c, this still does not provide a closed form expression for v. However, the above free boundary value problem can be expressed in variational form and that can be used as the basis for a numerical method for approximating v. See [**28**], Section 2.8, for more discussion of this and other methods for approximating v.

4.10. Exercises

For the following exercises, you should assume that the primary market model is given by the Black-Scholes model.

1. Verify that in the Example in Section 4.2, ϕ is not an admissible strategy.

2. Suppose that time is measured in years and that the riskless interest rate is 5% per year (compounded continuously). Assume that the stock moves according to a geometric Brownian motion with drift parameter $\mu = 0.15$, volatility parameter $\sigma = 0.20$ and a current price of \$10.

(a) Consider a European call option on the stock with an exercise price of \$12 and an expiration date of six months from now. What is the arbitrage free price for this option now? Describe a replicating strategy for this option.

(b) What is the current arbitrage free price for a European contingent claim that pays \$1 if the stock price is more than \$10 six months from now and pays off zero otherwise?

(c) By numerically approximating solutions of stochastic differential equations, or otherwise, estimate the current arbitrage free price of a European contingent claim called an *Asian* option that has a value at its expiration six months from now of

$$\left(2\int_0^{0.5} S_t\,dt - K\right)^+, \tag{4.93}$$

where S_t denotes the stock price t years from now and $K = \$12$.

3. Verify that C_0 given by the formula (4.62) is a strictly increasing function of σ for fixed K and T.

4. Consider a European call option with a strike price of \$70 and an expiration date of one month from now. The option is based on a stock with a current price of \$82. Suppose that the current price of the option is \$14. Assuming that the risk free interest rate is 6% per year, estimate the implied volatility of the stock. Please indicate the units in which your answer is measured. (You may use Newton's method or some other suitable numerical scheme. You will also need to use a reasonable approximation for the number of trading days between now and a month from now. Please use 21 days or 1/12 of a year for this.)

5. This exercise requires you to obtain current prices for options on IBM stock and historical data for the price of IBM stock. Such information can be readily obtained for example from the Yahoo Finance web site.

(a) For this part of the exercise, you will need a list of current prices for call options on IBM stock. Consider the call options on IBM stock that expire next month. These all expire on the same day of that month but may have differing strike prices and differing initial prices. For each of these options, assuming a Black-Scholes model and that these are European options, estimate the implied volatility. (Assume that the 3-month T-bill rate is the risk free interest rate.) Make a graph of implied volatility versus strike price from these calculations.

(b) This part of the exercise requires you to obtain historical data on the daily closing price of IBM stock. Assuming the Black-Scholes model, consider the approximation for the stock price S over the interval $[0, T]$ given by

$$S_{t+\Delta t} - S_t = \mu S_t \Delta t + \sigma S_t \Delta_t W, \tag{4.94}$$

where $\Delta t = T/n$ (some positive integer n), t is a multiple of Δt, $\Delta_t W = W_{t+\Delta t} - W_t$ is an increment of the Brownian motion W that is normally distributed with mean 0 and variance Δt. Here

the natural choice for Δt, given daily closing price data, is one day. (In our time line, we ignore non-trading days such as weekends and holidays.) For $i = 1, \ldots, n \equiv \frac{T}{\Delta t}$, let

$$X_i = \frac{S_{i\Delta t} - S_{(i-1)\Delta t}}{S_{(i-1)\Delta t}}. \tag{4.95}$$

Then, the usual estimators for μ and σ^2 are given by

$$\bar{\mu} = \frac{1}{T}\sum_{i=1}^{n} X_i, \qquad \bar{\sigma}^2 = \frac{1}{(n-1)\Delta t}\sum_{i=1}^{n}(X_i - \bar{\mu}\Delta t)^2. \tag{4.96}$$

Apply the above to the closing stock price data for IBM stock over a one-month period ending today. Compare the estimate $\bar{\sigma}$ of the volatility obtained in this way with the implied volatility estimates obtained from the option prices.

6. Let $X \in \mathcal{F}_T$ represent the final value of a European contingent claim and assume that $E^{P^*}[|X|] < \infty$. Verify that strategies of the form (4.46) satisfy the conditions (i)–(v) of Section 4.5 when the price process C for the European contingent claim is a continuous modification of $\{e^{rt}E^{P^*}[X^*|\mathcal{F}_t], t \in [0, T]\}$.

7. In a manner similar to that in Section 4.6, derive formulas for the arbitrage free price process and a replicating strategy associated with a European put option having a strike price of K and an expiration time of T.

8. Consider an American contingent claim and the associated process $\{U_t^*, t \in [0, T]\}$ as defined in Section 4.7. Prove that $\{U_t = e^{rt}U_t^*, t \in [0, T]\}$ is the unique arbitrage free price process for the American contingent claim. (For this, you will need to develop a notion of arbitrage for trading in stock, bond and the American contingent claim at all times $t \in [0, T]$. You should make sure to indicate this definition; cf. Section 4.5.)

9. Consider an American call option that can be traded only at time zero. Show that a buyer who purchases the American call option for its arbitrage free initial price and who does not exercise the option before time T will prevent the seller of the option from making a risk free profit (i.e., there will be no seller's arbitrage).

Chapter 5

Multi-dimensional Black-Scholes Model

In this chapter, we focus on a continuous financial market model that generalizes the Black-Scholes model of Chapter 4 in several directions. The model considered here has finitely many assets. One of these is a "riskless" money market asset and the others can be viewed as "risky" stocks. The price processes for the assets are given by an Itô process (cf. Appendix D) where the money market asset price process is non-decreasing with a zero volatility coefficient. Apart from this constraint, the drift and volatility coefficients of the Itô process can be stochastic and time dependent.

Following usage by some other authors [**32**], we call the (primary) financial market model considered here a multi-dimensional Black-Scholes model. This may be viewed as a continuous analogue of the finite market model studied in Chapter 3. One can consider more general financial market models using continuous semimartingales, or even general semimartingales with jumps, for the price processes. However, consideration of more general models would take us beyond the introductory level of this book. Moreover, the multi-dimensional Black-Scholes model considered here does allow us to illustrate how the fundamental theorems of asset pricing developed for the finite market model in Chapter 3 can be extended to a class of continuous models.

We begin by defining the multi-dimensional Black-Scholes model and the notion of a self-financing trading strategy. There are several different ways of defining such a model depending on the degree of measurability and integrability that one assumes for the coefficients in the Itô process and for the trading strategies. Our integrability assumptions on the trading

strategies are similar to those used in Karatzas and Shreve [**28**], Section 1.2. This form will be useful when proving some results related to completeness of the model.

After defining the financial market model, we describe the first fundamental theorem of asset pricing. Extrapolating from the finite market model case, we would expect this to say essentially that absence of arbitrage opportunities in the market model is equivalent to the existence of a risk neutral probability. The precise formulation of a suitably weak notion of arbitrage and of a risk neutral probability (as an equivalent *local* martingale measure) are subtle in the continuous setting. Although versions of the first fundamental theorem of asset pricing go back to the seminal works of Harrison, Kreps and Pliska [**20, 21, 22, 29**], a full understanding of this equivalence in a very general setting for price processes was developed only in the 1990s. Here we state a form of the fundamental theorem of asset pricing that derives from a 1994 result of Delbaen and Schachermayer [**14**]. Following that, we characterize the form of equivalent local martingale measures for the multi-dimensional Black-Scholes model. This exploits the Itô process form of the asset price processes and the (local) martingale representation theorem for Brownian motion.

Assuming that the market model is viable, which by the first fundamental theorem amounts to assuming that there exists an equivalent local martingale measure, we then describe the second fundamental theorem of asset pricing. This asserts that all suitably integrable European contingent claims are replicable (i.e., the market is complete) if and only if there is a *unique* equivalent local martingale measure. Assuming a certain non-degeneracy (i.e., there are no more stocks than there are degrees of freedom in the underlying Brownian motion driving the asset price processes), we show that the market is complete if and only if there are exactly as many stocks as there are degrees of freedom in the driving Brownian motion and almost surely the volatility matrix for the stocks is invertible at almost every time. There are some subtleties associated with these results on completeness that relate to integrability assumptions on self-financing strategies (see Harrison and Pliska [**21**], Section 3.3, for one indication of this). Our use of a similar setup to that of Karatzas and Shreve [**28**] is convenient for handling these subtleties. For more general market models (not treated here), completeness need not imply uniqueness of an equivalent local martingale measure (cf. Artzner and Heath [**1**] for an example), and it becomes a challenge to formulate a suitable variant of the second fundamental theorem (cf. Jarrow, Jin and Madan [**26**] for an approach to this and for a description of some of the related history).

Assuming the market model is viable and complete, we illustrate how the unique arbitrage free price for a suitably integrable European contingent claim can be obtained. We conclude the chapter with some references to the literature on incomplete markets.

5.1. Preliminaries

We use a similar basic setup to that of the Black-Scholes model treated in Chapter 4, except that now we have a multi-dimensional Brownian motion as a driving noise source.

We consider a finite time interval $[0, T]$, for some $0 < T < \infty$, as the interval during which trading may take place. The Borel σ-algebra of subsets of $[0, T]$ will be denoted by $\mathcal{B}_T$.

We assume as given a complete probability space $(\Omega, \mathcal{F}, P)$ on which is defined a standard n-dimensional *Brownian motion* $W = \{W_t, t \in [0, T]\}$ for some positive integer n. In particular, $W = (W^1, \ldots, W^n)$ is an n-dimensional process defined on the time interval $[0, T]$; the one-dimensional coordinate processes $W^1, \ldots, W^n$ are mutually independent; and for each $i = 1, \ldots, n$, $W^i = \{W^i_t, t \in [0, T]\}$ is a standard one-dimensional Brownian motion. Let $\{\mathcal{F}_t, t \in [0, T]\}$ denote the standard filtration generated by the n-dimensional Brownian motion W under P (cf. Appendix D, Section D.1). It is well known that this filtration is right continuous; i.e., for each $t \in [0, T)$, $\mathcal{F}_t = \mathcal{F}_{t+} \equiv \cap_{s \in (t, T]} \mathcal{F}_s$ (cf. Chung [**9**], Section 2.3, Theorem 4). The σ-algebra $\mathcal{F}_0$ is trivial in the sense that any $\mathcal{F}_0$-measurable random variable is P-a.s. constant. All random variables considered in this chapter will be assumed to be defined on $(\Omega, \mathcal{F}_T)$, and so without loss of generality we assume that $\mathcal{F} = \mathcal{F}_T$. We shall frequently write $\{\mathcal{F}_t\}$ instead of the more cumbersome $\{\mathcal{F}_t, t \in [0, T]\}$. The filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ satisfies the usual conditions (cf. Appendix C). It is well known that under this assumption, every martingale has a modification whose paths are all right continuous with finite left limits (cf. Chung [**9**], Theorem 3, page 29, and Corollary 1, page 26). In fact, since the filtration is generated by the Brownian motion W, every (local) martingale has a continuous modification (cf. Theorems II.2.9, V.3.5 in Revuz and Yor [**35**]). Expectations with respect to P will be denoted by $E[\,\cdot\,]$ unless there is more than one probability measure under consideration, in which case we shall use $E^P[\,\cdot\,]$ in place of $E[\,\cdot\,]$.

In the following, we shall encounter integrals of the form $\int_0^t R_s\, ds$, for $t \in [0, T]$, where $R = \{R_t, t \in [0, T]\}$ is an m-dimensional, adapted process (for some $m \geq 1$) satisfying

(i) $R : [0,T]\times\Omega \to \mathbb{R}^m$ is $(\mathcal{B}_T\times\mathcal{F}_T)$-measurable, where $R(t,\omega) = R_t(\omega)$ for each $t \in [0,T]$ and $\omega \in \Omega$;

(ii) $\int_0^T |R_s|\, ds < \infty$ P-a.s., where $|x| = (\sum_{i=1}^m (x^i)^2)^{\frac{1}{2}}$ for each $x = (x^1, \ldots, x^m) \in \mathbb{R}^m$.

Given these assumptions, P-a.s., the integral $\int_0^t R_s\, ds$ is well defined pathwise (and componentwise) for all $t \in [0,T]$. For convenience, on an exceptional P-null set, we define these integrals for all $t \in [0,T]$ to be zero. One can show that this defines a continuous adapted process $\{\int_0^t R_s\, ds, t \in [0,T]\}$ (cf. Appendix D, Section D.3). Whenever we encounter such pathwise integrals, we shall always assume that they have been defined suitably on an exceptional null set.

5.2. Multi-dimensional Black-Scholes Model

Our multi-dimensional Black-Scholes model has $d+1$ assets where d is a strictly positive integer. One of the assets is viewed as a money market instrument that has a possibly stochastic and time-dependent rate of return. The other d assets are viewed as stocks.

The *money market* price process, denoted by $B = \{B_t, t \in [0,T]\}$, is a continuous process given by

$$B_t = \exp\left(\int_0^t r_s\, ds\right), \quad t \in [0,T], \tag{5.1}$$

where $r = \{r_t, t \in [0,T]\}$ is an adapted process taking values in $[0,\infty)$ such that

(i) $r : [0,T]\times\Omega \to [0,\infty)$ is $(\mathcal{B}_T\times\mathcal{F}_T)$-measurable where $r(t,\omega) = r_t(\omega)$ for each $t \in [0,T]$ and $\omega \in \Omega$, and

(ii) $\int_0^T r_t\, dt < \infty$ P-a.s.

Remark. The convention mentioned at the end of the previous section is used in defining $\{\int_0^t r_s\, ds, t \in [0,T]\}$ as a continuous adapted process.

We may think of r_t as the short-term interest rate for the money market instrument at time t. The process $r = \{r_t, t \in [0,T]\}$ is sometimes called the *rate of return process*. The money market process satisfies $B_0 = 1$, and the following dynamic equation holds P-a.s.:

$$B_t = 1 + \int_0^t r_s B_s\, ds, \quad t \in [0,T]. \tag{5.2}$$

Since B has continuous paths that are non-decreasing (as $r_t \geq 0$ for all t), the money market asset is sometimes referred to as a riskless asset, although its rate of return process r can be stochastic (in a non-anticipating manner). We shall use the money market asset as a *numeraire*; i.e., a reference price

process which is used to normalize the other price processes in forming discounted values. In the Black-Scholes model of Chapter 4, the bond played the role of numeraire.

We assume that for some fixed positive integer d there are d *stocks* with continuous, adapted price processes $S^i = \{S_t^i, t \in [0,T]\}$, $i = 1, \ldots, d$, that take values in $(0, \infty)$ and are such that for each i and $t \in [0,T]$,

$$S_t^i = S_0^i \exp\left(\int_0^t \left(\mu_s^i - \frac{1}{2}\sigma_s^i \cdot \sigma_s^i\right) ds + \int_0^t \sigma_s^i \cdot dW_s\right), \tag{5.3}$$

where S_0^i is a strictly positive constant and μ^i and σ^i are given such that μ^i is the i^{th} component of a d-dimensional process $\mu = \{\mu_t, t \in [0,T]\}$ and $\sigma^i = \{\sigma_t^i, t \in [0,T]\}$ is the n-dimensional process given by the i^{th} row of a $(d \times n)$-matrix-valued process $\sigma = \{\sigma_t, t \in [0,T]\}$, and μ and σ are assumed to satisfy the following:

(i) $\mu = \{\mu_t, t \in [0,T]\}$ is a d-dimensional, adapted process such that $\mu : [0,T] \times \Omega \to \mathbb{R}^d$ is $(\mathcal{B}_T \times \mathcal{F}_T)$-measurable where $\mu(t,\omega) = \mu_t(\omega)$ for all $t \in [0,T]$ and $\omega \in \Omega$, and $\int_0^T |\mu_t|\, dt < \infty$ P-a.s., where $|\mu_t| = (\sum_{i=1}^d (\mu_t^i)^2)^{\frac{1}{2}}$ for $t \in [0,T]$;

(ii) $\sigma = \{\sigma_t, t \in [0,T]\}$ is an adapted process taking values in the set of $d \times n$ matrices with real entries (denoted by $\mathbb{R}^{d\times n}$), $\sigma : [0,T] \times \Omega \to \mathbb{R}^{d\times n}$ is $(\mathcal{B}_T \times \mathcal{F}_T)$-measurable where $\sigma(t,\omega) = \sigma_t(\omega)$ for all $t \in [0,T]$ and $\omega \in \Omega$, and $\int_0^T |\sigma_t|^2\, dt < \infty$ P-a.s., where $|\sigma_t|^2 = \sum_{i=1}^d \sum_{j=1}^n |\sigma_t^{ij}|^2$ for $t \in [0,T]$ and σ^{ij} denotes the ij component of the $d \times n$-matrix-valued process σ.

These assumptions ensure that the exponent in (5.3) defines the i^{th} component of an Itô process driven by the Brownian motion W (cf. Appendix D, Section D.3):

$$X_t^i = \int_0^t \left(\mu_s^i - \frac{1}{2}\sigma_s^i \cdot \sigma_s^i\right) ds + \int_0^t \sigma_s^i \cdot dW_s, \quad t \in [0,T],$$

for $i = 1, \ldots, d$. Using Itô's formula (cf. Appendix D, Section D.4) on the Itô process with the function $f : \mathbb{R}^d \to \mathbb{R}$ given by $f(x) = \exp(x^i)$ for $x \in \mathbb{R}^d$, we see that for $i = 1, \ldots, d$, S^i satisfies the following dynamic equation P-a.s.:

$$S_t^i = S_0^i + \int_0^t S_s^i \mu_s^i\, ds + \int_0^t S_s^i \sigma_s^i \cdot dW_s, \quad t \in [0,T]. \tag{5.4}$$

Indeed, this shows that the d-dimensional stock price process $(S^1, \ldots, S^d)$ is itself an Itô process driven by the Brownian motion W. Moreover, the process $(B, S^1, \ldots, S^d)$ is a $(d+1)$-dimensional Itô process, where the first component is a non-decreasing process.

For brevity, it is sometimes useful to write the equations (5.2) and (5.4), satisfied by the asset price processes, in differential form:

$$\begin{aligned} dB_t &= r_t\, B_t\, dt, \quad t \in [0,T], && (5.5) \\ dS_t^i &= S_t^i\, \mu_t^i\, dt + S_t^i\, \sigma_t^i \cdot dW_t, \quad t \in [0,T],\ i = 1,\ldots,d. && (5.6) \end{aligned}$$

We note for future reference that the process $B^{-1} = \{B_t^{-1}, t \in [0,T]\}$ where

$$B_t^{-1} = \exp\left(-\int_0^t r_s\, ds\right), \quad t \in [0,T], \tag{5.7}$$

is also an Itô process which satisfies the following dynamic equation P-a.s. for all $t \in [0,T]$:

$$B_t^{-1} = 1 - \int_0^t r_s\, B_s^{-1}\, ds. \tag{5.8}$$

For convenience, we call the money market instrument asset 0 and define $S_t^0 = B_t$ for all $t \in [0,T]$. We shall henceforth use B and S^0 interchangeably and we let $S = (S^0, S^1, \ldots, S^d)$. We define the *discounted asset price process* $S^* = (S^{*,0}, S^{*,1}, \ldots, S^{*,d})$ by

$$S_t^{*,i} = \frac{S_t^i}{B_t} = S_t^i \exp\left(-\int_0^t r_s\, ds\right), \quad t \in [0,T],\ i = 0,1,\ldots,d. \tag{5.9}$$

In particular, $S^{*,0} \equiv 1$. Using Itô's formula (cf. Appendix D) for the $(d+1)$-dimensional Itô process (B^{-1}, S) with the function $f(x,y) = xy^i$ for $x \in \mathbb{R}$ and $y \in \mathbb{R}^d$, together with (5.4) and (5.8), we see that for $i = 1, \ldots, d$,

$$dS_t^{*,i} = S_t^{*,i}\,(\mu_t^i - r_t)\, dt + S_t^{*,i}\, \sigma_t^i \cdot dW_t, \quad t \in [0,T], \tag{5.10}$$

or in other words, P-a.s., for all $t \in [0,T]$,

$$S_t^{*,i} = S_0^{*,i} + \int_0^t S_s^{*,i}\,(\mu_s^i - r_s)\, ds + \int_0^t S_s^{*,i}\, \sigma_s^i \cdot dW_s. \tag{5.11}$$

A *trading strategy* is a $(d+1)$-dimensional process

$$\phi = \{\phi_t = (\phi_t^0, \phi_t^1, \ldots, \phi_t^d), t \in [0,T]\}$$

such that

(i) $\phi : [0,T] \times \Omega \to \mathbb{R}^{d+1}$ is $(\mathcal{B}_T \times \mathcal{F}_T)$-measurable where $\phi(t,\omega) = \phi_t(\omega)$ for $t \in [0,T]$ and $\omega \in \Omega$;

(ii) ϕ is adapted, i.e., $\phi_t \in \mathcal{F}_t$ for each $t \in [0,T]$;

(iii) (a) $\int_0^T |\phi_t \cdot S_t|\, r_t\, dt < \infty$ P-a.s.,
(b) $\int_0^T |\sum_{i=1}^d \phi_t^i\, S_t^i\, (\mu_t^i - r_t)|\, dt < \infty$ P-a.s.,
(c) $\int_0^T \left|\sum_{i=1}^d \phi_t^i\, S_t^i\, \sigma_t^i\right|^2 dt < \infty$ P-a.s.

These conditions ensure that the integrals used below in defining the notion of self-financing yield continuous adapted processes (after possibly defining them suitably on an exceptional P-null set). Following Karatzas and Shreve [**28**], we have used a minimal set of integrability conditions on ϕ to facilitate some later manipulations associated with establishing conditions for completeness of the market model (cf. Section 5.5).

Here ϕ_t^0 is interpreted as the number of shares of the money market instrument held at time t, and for $i = 1, \ldots, d$, ϕ_t^i is interpreted as the number of shares of stock i held at time t. The *value* at time $t \in [0, T]$ of the portfolio associated with ϕ is given by

$$V_t(\phi) = \phi_t \cdot S_t = \sum_{i=0}^{d} \phi_t^i S_t^i. \tag{5.12}$$

Remark. The σ-algebra $\mathcal{F}_0$ is trivial; i.e., it contains only sets of probability zero or one. It follows that the $\mathcal{F}_0$-measurable random variable $V_0(\phi)$ is constant P-a.s. Thus, without loss of generality, by redefining ϕ_0 on a P-null set if necessary, we may and do assume that $V_0(\phi)$ is identically constant.

We define the *discounted value process*

$$V_t^*(\phi) = \frac{V_t(\phi)}{B_t}, \quad t \in [0, T]. \tag{5.13}$$

Note that, since $B_0 = 1$, $V_0^*(\phi) = V_0(\phi)$.

A trading strategy ϕ is said to be *self-financing* if $V(\phi) = \{V_t(\phi), t \in [0, T]\}$ is a continuous, adapted process such that P-a.s., for all $t \in [0, T]$,

$$\begin{aligned} V_t(\phi) &= V_0(\phi) + \int_0^t \phi_s \cdot S_s r_s \, ds + \int_0^t \left(\sum_{i=1}^{d} \phi_s^i \, S_s^i \, (\mu_s^i - r_s) \right) ds \\ &\quad + \int_0^t \left(\sum_{i=1}^{d} \phi_s^i \, S_s^i \, \sigma_s^i \right) \cdot dW_s. \end{aligned} \tag{5.14}$$

As mentioned above, the measurability and integrability assumptions placed on a trading strategy ϕ ensure that the integrals in the expression above can be used to define continuous adapted processes. We note that (5.14) can be rewritten as follows, which shows that $V(\phi)$ is a one-dimensional Itô process:

$$\begin{aligned} V_t(\phi) &= V_0(\phi) + \int_0^t \left(\phi_s^0 S_s^0 \, r_s + \sum_{i=1}^{d} \phi_s^i \, S_s^i \, \mu_s^i \right) ds \\ &\quad + \int_0^t \left(\sum_{i=1}^{d} \phi_s^i \, S_s^i \, \sigma_s^i \right) \cdot dW_s. \end{aligned} \tag{5.15}$$

With a suitable splitting up of the integral with respect to S, one may formally interpret the self-financing condition as representing the condition:

$$dV_t(\phi) = \phi_t \cdot dS_t, \qquad t \in [0,T].$$

In other words, the self-financing condition requires that changes in the value of the portfolio result only from changes in the values of the assets.

Similar to Lemma 4.2.1 in Chapter 4, we can characterize self-financing in terms of the discounted value process as follows.

Lemma 5.2.1. *A trading strategy ϕ is self-financing if and only if $V^*(\phi)$ is a continuous, adapted process such that P-a.s., for each $t \in [0,T]$,*

$$\begin{aligned} V_t^*(\phi) &= V_0(\phi) + \int_0^t \left(\sum_{i=1}^d \phi_s^i \, S_s^{*,i} \, (\mu_s^i - r_s) \right) ds \\ &\quad + \int_0^t \left(\sum_{i=1}^d \phi_s^i \, S_s^{*,i} \, \sigma_s^i \right) \cdot dW_s. \end{aligned} \tag{5.16}$$

Remark. The integrals on the right side of equation (5.16) are well defined by the conditions on ϕ, since the additional multiplicative factor in the integrands provided by the positive continuous process B^{-1} is P-a.s. bounded away from zero and infinity.

One may formally interpret the condition (5.16) as requiring that

$$dV_t^*(\phi) = \phi_t \cdot dS_t^*, \quad t \in [0,T]. \tag{5.17}$$

Proof. Note that $V(\phi)$ is continuous and adapted if and only if $V^*(\phi)$ has this property. Thus it suffices to show that, assuming this property, (5.14) holds if and only if (5.16) holds. Just as with Lemma 4.2.1, the proof of this involves simple manipulations using stochastic calculus. We shall give the proof of the "only if" part; the proof of the "if part" is very similar.

Suppose that ϕ is a self-financing trading strategy. Then $V(\phi)$ is an Itô process that satisfies the dynamic equation (5.14). Thus, $(B^{-1}, V(\phi))$ is a two-dimensional Itô process; and applying Itô's formula to this process with the function $f(x,y) = xy$ for $(x,y) \in \mathbb{R}^2$, we obtain P-a.s., for each

$t \in [0, T]$,

$$\begin{aligned}
V_t^*(\phi) &\equiv B_t^{-1} V_t(\phi) \\
&= V_0(\phi) + \int_0^t V_s(\phi)\, dB_s^{-1} + \int_0^t B_s^{-1}\, dV_s(\phi) \\
&= V_0(\phi) - \int_0^t V_s(\phi)\, B_s^{-1}\, r_s\, ds + \int_0^t B_s^{-1} \phi_s \cdot S_s\, r_s\, ds \\
&\quad + \int_0^t B_s^{-1} \left(\sum_{i=1}^d \phi_s^i\, S_s^i\, (\mu_s^i - r_s) \right) ds \\
&\quad + \int_0^t B_s^{-1} \left(\sum_{i=1}^d \phi_s^i\, S_s^i\, \sigma_s^i \right) \cdot dW_s \\
&= V_0(\phi) + \int_0^t \left(\sum_{i=1}^d \phi_s^i\, S_s^{*,i}\, (\mu_s^i - r_s) \right) ds \\
&\quad + \int_0^t \left(\sum_{i=1}^d \phi_s^i\, S_s^{*,i}\, \sigma_s^i \right) \cdot dW_s,
\end{aligned}$$

where the first two integrals in the third line cancel one another because $V_s(\phi) = \phi_s \cdot S_s$, and we have used the fact that $S_s^{*,i} = B_s^{-1} S_s^i$. Thus, $V^*(\phi)$ satisfies (5.16) and it too is an Itô process. □

We note that the right side of equation (5.16) only involves the discounted stock price processes, $S^{*,1}, \ldots, S^{*,d}$. Indeed, this observation enables us to specify a self-financing strategy simply by specifying the components corresponding to the stocks. More precisely, let $\hat{S}^* = (S^{*,1}, \ldots, S^{*,d})$. Then we have the following.

Lemma 5.2.2. *Suppose that* $\hat{\phi} = \{\hat{\phi}_t = (\hat{\phi}_t^1, \ldots, \hat{\phi}_t^d), t \in [0, T]\}$ *is a* d*-dimensional process such that*

(i) $\hat{\phi} : [0, T] \times \Omega \to \mathbb{R}^d$ *is* $(\mathcal{B}_T \times \mathcal{F}_T)$*-measurable where* $\hat{\phi}(t, \omega) = \hat{\phi}_t(\omega)$ *for* $t \in [0, T]$ *and* $\omega \in \Omega$;

(ii) $\hat{\phi}$ *is adapted, i.e.,* $\hat{\phi}_t \in \mathcal{F}_t$ *for each* $t \in [0, T]$;

(iii) (a) $\int_0^T |\sum_{i=1}^d \hat{\phi}_t^i\, S_t^{*,i}\, (\mu_t^i - r_t)|\, dt < \infty$ P*-a.s.*,
(b) $\int_0^T \left|\sum_{i=1}^d \hat{\phi}_t^i\, S_t^{*,i}\, \sigma_t^i\right|^2 dt < \infty$ P*-a.s.*

Let $G^*(\hat{\phi}) = \{G_t^*(\hat{\phi}), t \in [0,T]\}$ be a continuous, adapted process such that P-a.s., for all $t \in [0,T]$,

$$G_t^*(\hat{\phi}) = \int_0^t \left(\sum_{i=1}^d \hat{\phi}_s^i \, S_s^{*,i} \, (\mu_s^i - r_s) \right) ds + \int_0^t \left(\sum_{i=1}^d \hat{\phi}_s^i \, S_s^{*,i} \, \sigma_s^i \right) \cdot dW_s.$$

Given a constant V_0, let

$$\phi_t^0 = V_0 + G_t^*(\hat{\phi}) - \hat{\phi}_t \cdot \hat{S}_t^*, \quad t \in [0,T].$$

Then,

$$\phi = (\phi^0, \hat{\phi}^1, \ldots, \hat{\phi}^d)$$

is a self-financing trading strategy with discounted value process given by

$$V_t^*(\phi) = V_0 + G_t^*(\hat{\phi}), \quad t \in [0,T].$$

Proof. It follows from the measurability properties of $\hat{\phi}$ and $G^*(\hat{\phi})$ that ϕ satisfies the first two measurability properties (i) and (ii) required of a trading strategy. Furthermore, the integrability conditions (iii)(b) and (iii)(c) required of a trading strategy are implied by the conditions imposed on $\hat{\phi}$. Note for this that since B is a continuous strictly positive process, each path is bounded away from zero and infinity and so the conditions in (iii) above, imposed on $\hat{\phi}$, hold if and only if they hold with S in place of S^*.

For the integrability condition (iii)(a) required of a trading strategy ϕ, we note that by the definition of ϕ^0, for each $t \in [0,T]$,

$$\begin{aligned} \phi_t \cdot S_t &= B_t(\phi_t \cdot S_t^*) = B_t(\phi_t^0 + \hat{\phi}_t \cdot \hat{S}_t^*) \\ &= B_t(V_0 + G_t^*(\hat{\phi})), \end{aligned}$$

and so P-a.s., for all $t \in [0,T]$,

$$\int_0^t |\phi_s \cdot S_s| \, r_s \, ds \le \sup_{s \in [0,T]} |B_s(V_0 + G_s^*(\hat{\phi}))| \left(\int_0^T r_s \, ds \right) < \infty,$$

by the integrability assumptions on r and since $\{B_s(V_0 + G_s^*(\hat{\phi})), s \in [0,T]\}$ is continuous and hence bounded pathwise. Thus, ϕ is a trading strategy. The value process for ϕ is given by

$$V_t(\phi) = \phi_t \cdot S_t = B_t(V_0 + G_t^*(\hat{\phi})), \quad t \in [0,T],$$

and the discounted value process is given by

$$V_t^*(\phi) = V_0 + G_t^*(\hat{\phi}), \quad t \in [0,T],$$

which is continuous and adapted. From the definition of $G^*(\hat{\phi})$ and the fact that $\hat{\phi}^i = \phi^i$ for $i = 1, \ldots, d$, it follows that $V^*(\phi)$ satisfies (5.16). Hence, by Lemma 5.2.1, ϕ is self-financing. □

An *arbitrage opportunity* is a self-financing trading strategy ϕ such that

$$V_0(\phi) = 0, \quad V_T(\phi) \geq 0, \quad \text{and} \quad P(V_T(\phi) > 0) > 0. \tag{5.18}$$

Note that the same definition results if the value process $V(\phi)$ is replaced by the discounted value process $V^*(\phi)$ in (5.18) above.

5.3. First Fundamental Theorem of Asset Pricing

Based on the first fundamental theorem of asset pricing for the finite market model described in Chapter 3, one might be tempted to conjecture the following (false) "folk" theorem: "In the multi-dimensional Black-Scholes model, there exists a risk neutral probability if and only if there are no arbitrage opportunities." To obtain a rigorous version of this statement requires refinement of the notion of no arbitrage and careful specification of what one means by a risk neutral probability.

In the generality of the multi-dimensional Black-Scholes model considered here, the appropriate notion of risk neutral probability involves a change of probability such that, under a probability measure equivalent to P, the discounted asset price process is a *local* martingale. Recall from Appendix C that given a probability measure Q defined on $(\Omega, \mathcal{F})$, an m-dimensional process (where m is a positive integer), $M = \{M_t, t \in [0,T]\}$ defined on the filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$, is a *local martingale* under Q if M is an adapted process and there is a sequence of stopping times $\{\tau_k\}_{k=1}^{\infty}$ such that $\tau_k \leq \tau_{k+1}$ for each k, $\lim_{k\to\infty} \tau_k = \infty$ Q-a.s., and for each k,

$$M^k \equiv \{M_{t\wedge\tau_k}, \mathcal{F}_t, t \in [0,T]\}$$

is an m-dimensional martingale under Q (i.e., each component of M^k is a one-dimensional martingale under Q). Further recall from Chapter 4 that two probability measures Q and $\tilde{Q}$ on $(\Omega, \mathcal{F})$ are equivalent (or mutually absolutely continuous) provided for each $A \in \mathcal{F}$,

$$Q(A) = 0 \quad \text{if and only if} \quad \tilde{Q}(A) = 0; \tag{5.19}$$

i.e., Q and $\tilde{Q}$ have the same sets of probability zero.

Definition 5.3.1. *An equivalent local martingale measure (abbreviated as ELMM) is a probability measure P^* on $(\Omega, \mathcal{F})$ such that P^* is equivalent to P and $\{S_t^*, \mathcal{F}_t, t \in [0,T]\}$ is a local martingale under P^*.*

Remark. In the context of this chapter, we shall also refer to an equivalent local martingale measure as a *risk neutral probability*.

The following terminology and notation is needed to define an appropriate notion of no arbitrage for the multi-dimensional Black-Scholes model where the asset price processes are given by an Itô process. For this, recall that we say a random variable is bounded if it is bounded P-a.s. and (in)equalities between random variables hold P-a.s. In particular, two random variables are considered to be equal if they are equal P-a.s. (Sometimes we shall include the P-a.s. for emphasis, but not always.)

Definition 5.3.2. *A tame strategy is a self-financing trading strategy ϕ for which there is a constant $c > 0$ such that P-a.s.,*

$$V_t^*(\phi) \geq -c \quad \text{for all } t \in [0, T].$$

Let

$$L = \{V_T^*(\phi) : \phi \text{ is a tame strategy and } V_0(\phi) = 0 \ P\text{-}a.s.\}.$$

Thus, L is the set of discounted terminal values for portfolios that are achievable using tame strategies with zero initial value. We let D denote the set of bounded random variables that can be dominated by an element of L. That is, D is the set of bounded random variables such that there exists a tame strategy with an initial value of zero whose terminal discounted value is at least as large as that of the given random variable. If there is a non-negative random variable X in D satisfying $P(X > 0) > 0$, then there is an arbitrage opportunity using a tame strategy. A slightly more general notion of arbitrage opportunity is needed to yield a satisfactory first fundamental theorem. This is provided by considering the closure of D. Let $\overline{D}$ denote the closure of D in the space $L^\infty(\Omega, \mathcal{F}, P)$ of bounded, real-valued random variables defined on $(\Omega, \mathcal{F}, P)$ endowed with its usual essential supremum norm $\|\cdot\|_\infty$.

Definition 5.3.3. *The multi-dimensional Black Scholes model satisfies the "No Free Lunch with Vanishing Risk" (NFLVR) condition if and only if the only non-negative random variable in $\overline{D}$ is the zero random variable.*

The condition NFLVR implies that there is no sequence of tame strategies $\{\phi^n\}$ with zero initial values such that the discounted final payoffs satisfy $V_T^*(\phi^n) \geq -\frac{1}{n}$ P-a.s. for each n, and $Y \equiv \liminf_{n\to\infty} V_T^*(\phi^n)$ satisfies $P(Y > 0) > 0$.

The following is a statement of the *first fundamental theorem of asset pricing* in the context of our multi-dimensional Black-Scholes model. A proof of this result in a somewhat more general context (for locally bounded semimartingales) can be found in Delbaen and Schachermayer [**14**] (cf. Corollary 1.2 of Section 4). Their proof uses heavy machinery from stochastic processes and functional analysis. We refer the interested reader to that paper for the proof and also for a description of related forms of the result and for some

of the history associated with this topic. (In fact, the "only if" part of the theorem can be proved using the results on equivalent local martingale measures derived in Section 5.4 below. See the Exercises at the end of this chapter.)

Theorem 5.3.4. *(First Fundamental Theorem of Asset Pricing) An equivalent local martingale measure exists for the multi-dimensional Black-Scholes model if and only if there is "No Free Lunch with Vanishing Risk" (NFLVR).*

Our main interest in the preceding result is that it provides motivation for restricting to models for which there is at least one equivalent local martingale measure. In the next section we investigate the form of such measures. For later use, we say that the multi-dimensional Black-Scholes model is *viable* if and only if it satisfies the condition of "No Free Lunch with Vanishing Risk". By the theorem above, this may be expressed by requiring that there is at least one equivalent local martingale measure.

5.4. Form of Equivalent Local Martingale Measures

In the remaining sections of this chapter, since we shall be dealing with more than one probability measure on $(\Omega, \mathcal{F})$, when taking expectations we shall explicitly indicate the relevant probability measure. Recall that we are assuming that $\mathcal{F} = \mathcal{F}_T$.

In the first theorem below, we characterize the probability measures on $(\Omega, \mathcal{F})$ that are equivalent to P.

Theorem 5.4.1. *(Equivalent Probability Measures) Q is a probability measure equivalent to P on $(\Omega, \mathcal{F})$ if and only if there is an n-dimensional, adapted process $\theta = \{\theta_t, t \in [0,T]\}$ satisfying*

(i) $\theta : [0,T] \times \Omega \to \mathbb{R}^n$ *is* $(\mathcal{B}_T \times \mathcal{F}_T)$*-measurable, where* $\theta(t,\omega) = \theta_t(\omega)$ *for* $t \in [0,T]$ *and* $\omega \in \Omega$*;*

(ii) $\int_0^T |\theta_t|^2\, dt < \infty$ P*-a.s.,*

such that

$$\Lambda_t = \exp\left(- \int_0^t \theta_s \cdot dW_s - \frac{1}{2} \int_0^t |\theta_s|^2\, ds \right), \quad t \in [0,T], \tag{5.20}$$

defines a continuous martingale under P*, and*

$$Q(A) = E^P[1_A \Lambda_T] \quad \textit{for all} \quad A \in \mathcal{F}. \tag{5.21}$$

Remark. As usual, to ensure that Λ is well defined and has continuous paths everywhere, on an exceptional P-null set where $\int_0^T |\theta_t|^2\, dt$ may not be finite, we define the integrals in the exponent of Λ_t to be zero for all t, and then Λ_t is one for all $t \in [0,T]$. A well-known sufficient condition for

the process Λ to be a martingale under P given that θ is an n-dimensional process satisfying conditions (i)–(ii) of Theorem 5.4.1 is called *Novikov's criterion*. This criterion requires that

$$E^P\left[\exp\left(\frac{1}{2}\int_0^T |\theta_t|^2\,dt\right)\right] < \infty. \tag{5.22}$$

Proof. If θ is an n-dimensional adapted process satisfying conditions (i)–(ii) of the theorem, then the stochastic integral process $\{\int_0^t \theta_s \cdot dW_s, t \in [0,T]\}$ is well defined as a continuous local martingale under P. As described in the remark above, up to indistinguishability, $\{\int_0^t |\theta_s|^2 ds, t \in [0,T]\}$ defines a continuous, adapted process. Hence, Λ can be defined as a continuous, adapted process that satisfies $\Lambda_t > 0$ for all $t \in [0,T]$. In fact, under P, by applying Itô's formula to the Itô process

$$X_t = -\frac{1}{2}\int_0^t |\theta_s|^2\,ds - \int_0^t \theta_s \cdot dW_s,\ t \in [0,T]$$

and the function $f : \mathbb{R} \to (0,\infty)$ given by $f(x) = \exp(x)$, we see that P-a.s., for all $t \in [0,T]$,

$$\begin{aligned}
\Lambda_t &= \exp(X_t) \\
&= 1 + \int_0^t \Lambda_s\,dX_s + \frac{1}{2}\int_0^t \Lambda_s\,d[X]_s \\
&= 1 - \int_0^t \Lambda_s\,\theta_s \cdot dW_s - \frac{1}{2}\int_0^t \Lambda_s\,|\theta_s|^2\,ds \\
&\qquad + \frac{1}{2}\int_0^t \Lambda_s|\theta_s|^2\,ds \\
&= 1 - \int_0^t \Lambda_s\,\theta_s \cdot dW_s.
\end{aligned} \tag{5.23}$$

Since Λ is a continuous, adapted process, $\Lambda\theta = \{\Lambda_t\theta_t, t \in [0,T]\}$ is adapted and satisfies the same conditions (i)–(ii) as θ; hence the stochastic integral in the last line above is well defined as a continuous local martingale under P, and so Λ is a continuous local martingale under P. If in fact Λ is a martingale under P, then

$$E^P[\Lambda_T] = E^P[\Lambda_0] = 1$$

and (5.21) defines a probability measure Q on $(\Omega, \mathcal{F})$ and, since $\Lambda_T > 0$, this probability measure is equivalent to P. This completes the proof of the "if" part of the theorem.

To prove the "only if" part of the theorem, suppose that Q is a probability measure on $(\Omega, \mathcal{F})$ that is equivalent to P. Let

$$\zeta = \frac{dQ}{dP} \quad \text{on } \mathcal{F}; \tag{5.24}$$

i.e., ζ is the Radon-Nikodym derivative of Q with respect to P. (As with conditional expectations, such a random variable is uniquely defined only up to P-null sets.) Thus, ζ is a non-negative $\mathcal{F}_T$-measurable (recall that $\mathcal{F}_T = \mathcal{F}$) random variable such that

$$Q(A) = E^P[\zeta 1_A] \quad \text{for each} \quad A \in \mathcal{F}. \tag{5.25}$$

In particular, $E^P[\zeta] = Q(\Omega) = 1$ and so ζ is integrable with respect to P. Furthermore, since Q is *equivalent* to P, we have $\zeta > 0$ P-a.s. We define a martingale $\Lambda = \{\Lambda_t, t \in [0,T]\}$ under P by setting

$$\Lambda_t = E^P[\zeta \mid \mathcal{F}_t] \quad \text{for} \quad t \in [0,T].$$

Since, for each $t \in [0,T]$, the conditional expectation above is defined only up to a P-null set and every martingale with respect to the Brownian filtration $\{\mathcal{F}_t\}$ under P has a continuous modification, without loss of generality, we may and do assume that all of the paths of the martingale Λ are continuous. Note that since ζ is measurable with respect to $\mathcal{F} = \mathcal{F}_T$, we have $\Lambda_T = \zeta$ P-a.s. Also, P-a.s.,

$$\Lambda_0 = E^P[\zeta \mid \mathcal{F}_0] = E^P[\zeta] = 1,$$

and since $\zeta \geq 0$ P-a.s., we have for each fixed $t \in [0,T]$,

$$\Lambda_t = E^P[\zeta \mid \mathcal{F}_t] \geq 0 \quad P - a.s.$$

In fact we can show that P-a.s., $\Lambda_t > 0$ for all $t \in [0,T]$, as follows.

Consider the stopping time

$$\tau = \inf\{t \in [0,T] : \Lambda_t = 0\},$$

where the infimum of the empty set is defined to be ∞. Then by Doob's stopping theorem for martingales (cf. Appendix C),

$$E^P[\Lambda_{T\wedge\tau}] = E^P[\Lambda_0] = 1.$$

Splitting up the expectation on the left according to whether $\tau = \infty$ or $\tau \leq T$, we have

$$E^P[\Lambda_T 1_{\{\tau=\infty\}}] + E^P[\Lambda_\tau 1_{\{\tau\leq T\}}] = 1.$$

The second expectation above is zero since $\Lambda_\tau = 0$ on $\{\tau \leq T\}$. Hence,

$$E^P[\Lambda_T 1_{\{\tau=\infty\}}] = 1 = E^P[\Lambda_0] = E^P[\Lambda_T],$$

where we have used the martingale property of Λ to obtain the last equality. By subtraction we find that

$$E^P[\Lambda_T 1_{\{\tau\leq T\}}] = 0,$$

and hence since $\Lambda_T = \zeta$ P-a.s., we have

$$Q(\tau \leq T) = E^P[\zeta\, 1_{\{\tau\leq T\}}] = 0;$$

and since P is equivalent to Q, it follows that

$$P(\tau \leq T) = 0.$$

Hence, since $\Lambda_0 = 1 > 0$ P-a.s. and Λ has continuous paths, it follows that P-a.s.,

$$\Lambda_t > 0 \quad \text{for all } t \in [0, T], \tag{5.26}$$

as desired.

We now apply the martingale representation theorem to Λ (cf. Theorem D.6.1 of Appendix D). Since, under P, Λ is a continuous martingale with respect to the standard filtration $\{\mathcal{F}_t\}$ associated with the n-dimensional Brownian motion W, by the martingale representation theorem, there is an adapted n-dimensional process $\eta = \{\eta_t, t \in [0, T]\}$ such that $\eta : [0, T] \times \Omega \to \mathbb{R}^n$ is $(\mathcal{B}_T \times \mathcal{F}_T)$-measurable, $\int_0^T |\eta_s|^2 \, ds < \infty$ P-a.s., and

$$\Lambda_t = \Lambda_0 + \int_0^t \eta_s \cdot dW_s \quad \text{for all } \ t \in [0, T], \quad P - a.s.$$

Let N be a P-null set such that (5.26) holds everywhere on $\Omega \setminus N$. Define

$$\theta_t(\omega) = \begin{cases} -\frac{\eta_t(\omega)}{\Lambda_t(\omega)} & \text{for all } \ t \in [0, T], \ \omega \in \Omega \setminus N, \\ 0 & \text{for all } \ t \in [0, T], \quad \omega \in N. \end{cases} \tag{5.27}$$

Then, P-a.s.,

$$-\Lambda_t \theta_t = \eta_t \quad \text{for all } t \in [0, T].$$

By the continuity properties of Λ and the fact that (5.26) holds on $\Omega \setminus N$, the process θ inherits the properties of η; i.e., $\theta = \{\theta_t, t \in [0, T]\}$ is an adapted process where $\theta : [0, T] \times \Omega \to \mathbb{R}^n$ is $(\mathcal{B}_T \times \mathcal{F}_T)$-measurable and $\int_0^T |\theta_t|^2 dt < \infty$ P-a.s. Consequently, the stochastic integral process

$$\int_0^t \theta_s \cdot dW_s = \sum_{i=1}^n \int_0^t \theta_s^i \, dW_s^i, \quad t \in [0, T],$$

is well defined as a continuous local martingale under P (cf. Appendix D). Let N' denote a P-null set such that $\int_0^T |\theta_s|^2 ds < \infty$ on $\Omega \setminus N'$. Define a continuous adapted process $\rho = \{\rho_t, t \in [0, T]\}$ by

$$\rho_t = \begin{cases} \exp\left(-\int_0^t \theta_s \cdot dW_s - \frac{1}{2} \int_0^t |\theta_s|^2 \, ds \right), & \text{for } \ t \in [0, T], \text{ on } \Omega \setminus N', \\ 1, & \text{for } \ t \in [0, T], \text{ on } N'. \end{cases}$$

Then, $\rho_0 = 1$, and by Itô's formula, ρ satisfies the following stochastic equation P-a.s.:

$$\rho_t = 1 - \int_0^t \rho_s \, \theta_s \cdot dW_s, \quad t \in [0, T]. \tag{5.28}$$

On the other hand, $\Lambda_0 = 1$ P-a.s., and Λ satisfies the following stochastic equation P-a.s.:

$$\Lambda_t = 1 + \int_0^t \eta_s \cdot dW_s = 1 - \int_0^t \Lambda_s \theta_s \cdot dW_s, \quad t \in [0,T]. \tag{5.29}$$

Thus, Λ and ρ satisfy the same stochastic equation with the same initial condition. We will show that in fact P-a.s.,

$$\rho_t = \Lambda_t \quad \text{for all} \quad t \in [0,T]. \tag{5.30}$$

Indeed, since P-a.s., $\Lambda_t > 0$ and $\rho_t > 0$ for all $t \in [0,T]$, one can apply Itô's formula (cf. Section D.4 in Appendix D) to the two-dimensional Itô process (Λ, ρ) and the function $f : (0,\infty)^2 \to \mathbb{R}$ given by $f(x,y) = x/y$ to yield P-a.s. for all $t \in [0,T]$,

$$\begin{aligned}
\frac{\Lambda_t}{\rho_t} &= \frac{\Lambda_0}{\rho_0} + \int_0^t \rho_s^{-1}\, d\Lambda_s - \int_0^t \rho_s^{-2} \Lambda_s\, d\rho_s \\
&\quad + \frac{1}{2}\left\{ \int_0^t \frac{2}{\rho_s^3} \Lambda_s\, d[\rho]_s - 2\int_0^t \rho_s^{-2}\, d\langle \Lambda, \rho\rangle_s \right\} \\
&= \frac{\Lambda_0}{\rho_0} - \int_0^t \rho_s^{-1} \Lambda_s \theta_s \cdot dW_s + \int_0^t \rho_s^{-2} \Lambda_s \rho_s \theta_s \cdot dW_s \\
&\quad + \int_0^t \rho_s^{-3} \Lambda_s \rho_s^2 |\theta_s|^2\, ds - \int_0^t \rho_s^{-2} \rho_s \Lambda_s |\theta_s|^2\, ds \\
&= 1.
\end{aligned}$$

Here, $[\rho]$ denotes the quadratic variation process for ρ which, using the stochastic differential equation satisfied by ρ, can be seen to be given P-a.s. by

$$[\rho]_t = \int_0^t \rho_s^2 |\theta_s|^2\, ds, \quad t \in [0,T],$$

and $\langle \Lambda, \rho \rangle$ is the mutual variation (or covariation) process for Λ and ρ which, using the stochastic differential equations satisfied by Λ and ρ, can be seen to be given P-a.s. by

$$\langle \Lambda, \rho\rangle_t = \int_0^t \rho_s \Lambda_s |\theta_s|^2\, ds, \quad t \in [0,T].$$

To summarize, we have shown that P-a.s.,

$$\Lambda_t = \rho_t = \exp\left(-\int_0^t \theta_s \cdot dW_s - \frac{1}{2}\int_0^t |\theta_s|^2\, ds \right) \quad \text{for all } t \in [0,T], \tag{5.31}$$

where Λ is a continuous martingale with respect to P. Thus, P-a.s.,

$$\frac{dQ}{dP} = \zeta = \Lambda_T = \exp\left(-\int_0^T \theta_t \cdot dW_t - \frac{1}{2}\int_0^T |\theta_t|^2\, dt \right). \tag{5.32}$$

$\square$

The following lemma provides a useful characterization of local martingales under a probability measure equivalent to P. A proof of this can be found for example in Chung and Williams [**11**], Theorem 9.7.

Lemma 5.4.2. *Suppose that Q is a probability measure on $(\Omega, \mathcal{F})$ that is equivalent to P. Let $\theta = \{\theta_t, t \in [0,T]\}$ and $\Lambda = \{\Lambda_t, t \in [0,T]\}$ be as specified in Theorem 5.4.1. Then, a real-valued, right continuous adapted process $M = \{M_t, t \in [0,T]\}$ is a local martingale under Q if and only if $\{\Lambda_t M_t, t \in [0,T]\}$ is a local martingale under P.*

We now characterize the equivalent local martingale measures.

Theorem 5.4.3. *(Equivalent Local Martingale Measures) Q is an equivalent local martingale measure if and only if there is an n-dimensional, adapted process $\theta = \{\theta_t, t \in [0,T]\}$ such that*

(i) *θ satisfies conditions (i)–(ii) of Theorem 5.4.1,*

(ii) *(5.20) defines a continuous martingale under P, and Q is given by (5.21),*

(iii) *P-a.s.,*

$$\mu_t^i - r_t - \sigma_t^i \cdot \theta_t = 0 \quad \textit{for a.e. } t \in [0,T], \; i = 1, \ldots, d. \tag{5.33}$$

Remark. Here a.e. denotes "almost everywhere" with respect to Lebesgue measure on $[0,T]$. A process $\theta = \{\theta_t, t \in [0,T]\}$ satisfying all of the conditions in Theorem 5.4.3 is called a *market price of risk*. In general it need not be unique.

Proof. For a proof of the "only if" part of the theorem, suppose that Q is an equivalent local martingale measure. Since Q is equivalent to P, there is a process θ and an associated martingale $\Lambda = \{\Lambda_t, t \in [0,T]\}$ as described in Theorem 5.4.1, so that Λ_T yields the Radon-Nikodym derivative of Q with respect to P. By the local martingale property, S^* is a local martingale under Q. In fact, S^* has continuous paths, and so it is a continuous local martingale under Q. It follows from Lemma 5.4.2 that for each $i = 1, \ldots, d$, $\Lambda S^{*,i} = \{\Lambda_t S_t^{*,i}, t \in [0,T]\}$ is a local martingale under P, and hence $\Lambda S^* = (\Lambda S^{*,1}, \ldots, \Lambda S^{*,d})$ is a continuous d-dimensional local martingale under P.

For each $i = 1, \ldots, d$, we apply Itô's formula to the two-dimensional Itô process $(S^{*,i}, \Lambda)$ under P, and the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = xy$, $(x,y) \in \mathbb{R}^2$. Using the stochastic differential equations (5.11) and (5.23)

satisfied by $S^{*,i}$ and Λ, we have P-a.s. for all $t \in [0,T]$:

$$\begin{aligned}
\Lambda_t S_t^{*,i} &= \Lambda_0 S_0^{*,i} + \int_0^t \Lambda_s \, dS_s^{*,i} + \int_0^t S_s^{*,i} \, d\Lambda_s + \langle \Lambda, S^{*,i} \rangle_t \\
&= S_0^{*,i} + \int_0^t \Lambda_s \, S_s^{*,i} \, (\mu_s^i - r_s) \, ds + \int_0^t \Lambda_s \, S_s^{*,i} \, \sigma_s^i \cdot dW_s \\
&\quad - \int_0^t S_s^{*,i} \, \Lambda_s \, \theta_s \cdot dW_s - \int_0^t \Lambda_s \, S_s^{*,i} \, \sigma_s^i \cdot \theta_s \, ds \\
&= 1 + \int_0^t \Lambda_s \, S_s^{*,i} \, (\mu_s^i - r_s - \sigma_s^i \cdot \theta_s) \, ds \\
&\quad + \int_0^t \Lambda_s \, S_s^{*,i} \, (\sigma_s^i - \theta_s) \cdot dW_s. \qquad (5.34)
\end{aligned}$$

The equation above has a continuous local martingale (under P) on the left side. On the right side there are a sum of a continuous, adapted process whose paths are P-a.s. of bounded variation on $[0,T]$ and a continuous local martingale (under P) defined by the stochastic integral with respect to W. Rearranging to isolate the bounded variation process, we obtain a continuous local martingale (under P) equal to a continuous, adapted process whose paths are P-a.s. of bounded variation. This can happen only if both sides are P-a.s. identically constant (cf. [**11**], Corollary 4.5). Hence the continuous, adapted process whose paths are P-a.s. of bounded variation is P-a.s. identically constant. Thus, P-a.s.,

$$\int_0^t \Lambda_s \, S_s^{*,i} \, (\mu_s^i - r_s - \sigma_s^i \cdot \theta_s) \, ds = 0 \quad \text{for all} \quad t \in [0,T].$$

This implies that P-a.s.,

$$\Lambda_t \, S_t^{*,i} (\mu_t^i - r_t - \sigma_t^i \cdot \theta_t) = 0 \quad \text{for a.e. } t \in [0,T]. \qquad (5.35)$$

Now since P-a.s., $\Lambda_t > 0$ and $S_t^{*,i} > 0$ for all $t \in [0,T]$, it follows that P-a.s., θ satisfies (5.33). This completes the proof of the "only if" part of the theorem.

For a proof of the "if" part of the theorem, let θ be an n-dimensional, adapted process satisfying conditions (i)–(iii) of the theorem. Let $\Lambda = \{\Lambda_t, t \in [0,T]\}$ and Q be defined as in (5.20) and (5.21), respectively. Then, by Theorem 5.4.1, Q is equivalent to P, and it suffices to show that S^* is a local martingale under Q. By Lemma 5.4.2, for the latter, it suffices to show that $\{\Lambda_t S_t^{*,i}, t \in [0,T]\}$ is a local martingale under P for each $i = 1, \ldots, d$. In the same manner as for (5.34), since $(S^{*,i}, \Lambda)$ is a two-dimensional Itô process under P, by applying Itô's formula to this process and the function $f(x,y) = xy$, $(x,y) \in \mathbb{R}^2$, we have that for $i = 1, \ldots, d$, (5.34) holds P-a.s.

for all $t \in [0,T]$. Then using (5.33), this simplifies to

$$\Lambda_t S_t^{*,i} = 1 + \int_0^t \Lambda_s \, S_s^{*,i} \, (\sigma_s^i - \theta_s^i) \cdot dW_s.$$

Thus, we have that $\Lambda S^{*,i}$ is a constant plus a stochastic integral with respect to the Brownian motion W under P, and hence it is a local martingale under P. It then follows from Lemma 5.4.2 that $S^{*,i}$ is a local martingale under Q, as desired. □

Corollary 5.4.4. *Suppose that Q is an equivalent local martingale measure and that θ is an n-dimensional process as described in Theorem 5.4.3. Define the n-dimensional, continuous adapted process $\tilde{W} = \{\tilde{W}_t, t \in [0,T]\}$ such that*

$$\tilde{W}_t = W_t + \int_0^t \theta_s \, ds, \quad t \in [0,T]. \tag{5.36}$$

Then, on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, Q)$, $\tilde{W}$ is a standard n-dimensional Brownian motion martingale and Q-a.s., for $i = 1, \ldots, d$,

$$S_t^{*,i} = S_0^{*,i} + \int_0^t S_s^{*,i} \, \sigma_s^i \cdot d\tilde{W}_s, \quad t \in [0,T]. \tag{5.37}$$

Furthermore, for any self-financing trading strategy ϕ, the discounted value process $V^*(\phi)$ is a continuous local martingale under Q that Q-a.s. satisfies

$$V_t^*(\phi) = V_0(\phi) + \int_0^t \left(\sum_{i=1}^d \phi_s^i \, S_s^{*,i} \, \sigma_s^i \right) \cdot d\tilde{W}_s, \quad t \in [0,T]. \tag{5.38}$$

Remark. As usual, in the definition of $\tilde{W}$, we use a continuous, adapted representative of $\{\int_0^t \theta_s \, ds, t \in [0,T]\}$. The integrals in (5.37) and (5.38) are all well defined as stochastic integrals (yielding continuous local martingales) under Q.

Proof. By Girsanov's theorem (cf. Appendix D, Theorem D.5.1), $\tilde{W}$ is an n-dimensional Brownian motion martingale on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, Q)$. Now, under P, $\tilde{W}$ is an n-dimensional Itô process; and on substituting for W in terms of $\tilde{W}$ and θ in the the stochastic differential equation (5.11) satisfied by S^* and substituting the relation (5.33), we obtain P-a.s., for all $t \in [0,T]$ and $i = 1, \ldots, d$,

$$\begin{aligned} S_t^{*,i} &= S_0^{*,i} + \int_0^t S_s^{*,i} \left(\mu_s^i - r_s - \sigma_s^i \cdot \theta_s \right) ds + \int_0^t S_s^{*,i} \, \sigma_s^i \cdot d\tilde{W}_s \\ &= S_0^{*,i} + \int_0^t S_s^{*,i} \, \sigma_s^i \cdot d\tilde{W}_s. \end{aligned}$$

Note that, P-a.s., the integral $\int_0^t \sigma_s^i \cdot \theta_s \, ds$ is finite for all $t \in [0,T]$ by the Cauchy-Schwarz inequality and the integral assumptions on σ and θ. Also,

the integral with respect to $\tilde{W}$ can be defined as a P-almost sure limit of integrals with respect to $\tilde{W}$ against simple predictable integrands. Since Q is equivalent to P, the above relation holds Q-a.s., and under Q, the integral with respect to $\tilde{W}$ defines a continuous local martingale stochastic integral process with respect to the n-dimensional Brownian motion martingale $\tilde{W}$. This establishes (5.37).

By Lemma 5.2.1, for a self-financing trading strategy ϕ, (5.16) holds P-a.s., where on substituting the relation (5.33) and the definition of $\tilde{W}$, we obtain P-a.s., for all $t \in [0,T]$,

$$\begin{aligned} V_t^*(\phi) &= V_0(\phi) + \int_0^t \left(\sum_{i=1}^d \phi_s^i \, S_s^{*,i} \, \sigma_s^i \cdot \theta_s \right) ds \\ &\quad + \int_0^t \left(\sum_{i=1}^d \phi_s^i \, S_s^{*,i} \, \sigma_s^i \right) \cdot dW_s, \\ &= \int_0^t \left(\sum_{i=1}^d \phi_s^i \, S_s^{*,i} \, \sigma_s^i \right) \cdot d\tilde{W}_s. \end{aligned}$$

By similar reasoning to that in the paragraph above, since P is equivalent to Q, the above relation also holds Q-a.s., where the last expression is well defined as a continuous local martingale stochastic integral process under Q. □

Theorems 5.4.1 and 5.4.3 can be combined with the remark following Theorem 5.4.1 to yield the following result. For this, let $\mathbf{1}$ denote the d-dimensional vector of all ones.

Corollary 5.4.5. *Assume that $d = n$ and P-a.s., σ_t is invertible for a.e. $t \in [0,T]$. Suppose that $\theta = \{\theta_t, t \in [0,T]\}$ is an n-dimensional, adapted process such that P-a.s.,*

$$\sigma_t \theta_t = \mu_t - r_t \mathbf{1}, \quad \text{for a.e. } t \in [0,T]. \tag{5.39}$$

Further suppose that θ satisfies conditions (i)–(ii) of Theorem 5.4.1, as well as Novikov's criterion (5.22). Then, $\Lambda = \{\Lambda_t, t \in [0,T]\}$ defined by (5.20) in Theorem 5.4.1 is a martingale and Q defined by (5.21) is the unique equivalent local martingale measure.

Proof. The only detail requiring verification is the uniqueness of the ELMM. This follows from the fact that given an equivalent local martingale measure Q, there is always a θ as described in Theorem 5.4.3; and when $d = n$ and P-a.s., σ_t is invertible for a.e. $t \in [0,T]$, this relation determines $\theta : [0,T] \times \Omega \to \mathbb{R}^n$ up to sets of $(\lambda \times P)$-measure zero, where λ denotes Lebesgue measure on

$[0, T]$. This suffices for uniqueness up to indistinguishability of the associated process Λ given by (5.20). Hence Q (cf. (5.21)) is uniquely determined by σ, μ and r. □

5.5. Second Fundamental Theorem of Asset Pricing

Recall from the end of Section 5.3 that the multi-dimensional Black-Scholes model is *viable* if there is at least one equivalent local martingale measure (ELMM). We henceforth assume that the model is viable and let P^* be an equivalent local martingale measure.

Paralleling the definition of admissible strategies for the simple Black-Scholes model given in Chapter 4, we make the following definition for the multi-dimensional model considered here. In contrast to the situation for the simple model, here there may be more than one equivalent local martingale measure and so the set of admissible strategies can depend on the choice of that measure. Of course, if there is a unique ELMM, then the definition of admissible strategy is unequivocal.

An *admissible strategy* (relative to P^*) is a self-financing trading strategy ϕ such that the discounted value process $\{V_t^*(\phi), t \in [0, T]\}$ is a martingale under the equivalent local martingale measure P^*.

Given the form (5.38) of $V^*(\phi)$ under the ELMM P^*, a sufficient condition for a self-financing trading strategy ϕ to be admissible is that

$$E^{P^*}\left[\int_0^T \left|\sum_{i=1}^d \phi_s^i S_s^{*,i} \sigma_s^i\right|^2 ds\right] < \infty. \tag{5.40}$$

In this case, the discounted value process $V^*(\phi)$ will be an L^2-martingale under P^* (cf. Appendix D).

A *European contingent claim* is represented by an $\mathcal{F}_T$-measurable random variable X. We denote the discounted value of a European contingent claim X by

$$X^* = X/B_T = X/S_T^0.$$

A *replicating (or hedging) strategy* (relative to P^*) for a European contingent claim X is an admissible strategy (relative to P^*) such that $V_T(\phi) = X$, P^*-a.s.

Theorem 5.5.1. *(Second Fundamental Theorem of Asset Pricing) Suppose there exists an equivalent local martingale measure P^*. Then the following two conditions are equivalent.*

(i) *P^* is the unique equivalent local martingale measure.*

(ii) *Every $\mathcal{F}_T$-measurable random variable X satisfying $E^{P^*}[|X^*|] < \infty$ is replicable (relative to P^*).*

Proof. *(i) implies (ii):* This part of the proof uses a general result taken from Jacod and Yor [**25**]. We state the consequence of that result here without proof and refer the reader to that paper for details.

Suppose that P^* is the unique equivalent local martingale measure. Let θ and $\tilde{W}$ be as described in Corollary 5.4.4 with P^* in place of Q there. Let X be an $\mathcal{F}_T$-measurable random variable satisfying $E^{P^*}[|X^*|] < \infty$. Then, the conditional expectations

$$E^{P^*}[X^*|\mathcal{F}_t], \quad t \in [0,T],$$

define a martingale under P^*, and we may choose a modification that has right continuous paths with finite left limits. We denote such a representative by $M = \{M_t, t \in [0,T]\}$. It then follows from a result in the general theory of martingale representations (cf. Jacod and Yor [**25**], Proposition 2.4) that there is an adapted d-dimensional process $\eta = \{\eta_t, t \in [0,T]\}$ satisfying

(a) $\eta : [0,T] \times \Omega \to \mathbb{R}^d$ is $(\mathcal{B}_T \times \mathcal{F}_T)$-measurable where $\eta(t,\omega) = \eta_t(\omega)$ for $t \in [0,T]$ and $\omega \in \Omega$, and

(b) P^*-a.s.,

$$\sum_{i=1}^{d} \int_0^T |\eta_t^i\, S_t^{*,i} \sigma_t^i|^2\, dt < \infty, \tag{5.41}$$

and such that P^*-a.s., for all $t \in [0,T]$,

$$\begin{aligned} M_t &= M_0 + \int_0^t \eta_s \cdot dS_s^* \\ &= M_0 + \sum_{i=1}^{d} \int_0^t \eta_s^i\, dS_s^{*,i} \\ &= M_0 + \sum_{i=1}^{d} \int_0^t \eta_s^i\, S_s^{*,i} \sigma_s^i \cdot d\tilde{W}_s \\ &= M_0 + \int_0^t \sum_{i=1}^{d} \left(\eta_s^i\, S_s^{*,i} \sigma_s^i\right) \cdot d\tilde{W}_s. \end{aligned} \tag{5.42}$$

In other words, we can represent M as a stochastic integral with respect to the local martingale process S^*. It follows from this representation that M has continuous paths P^*-a.s. Note that by the Cauchy-Schwarz inequality, for each $t \in [0,T]$,

$$\left|\sum_{i=1}^{d} \eta_t^i\, S_t^{*,i} \sigma_t^i\right|^2 \le d \sum_{i=1}^{d} \left|\eta_t^i\, S_t^{*,i} \sigma_t^i\right|^2. \tag{5.43}$$

It follows from this and the conditions satisfied by η, and since P^* is equivalent to P, that $\hat{\phi} = \eta$ satisfies conditions (i)–(ii) and (iii)(b) of Lemma 5.2.2. In fact, condition (iii)(a) there also holds because θ satisfies the conditions in Theorem 5.4.3. In particular, P-a.s.,

$$\begin{aligned}\int_0^T \left| \sum_{i=1}^d \eta_t^i \, S_t^{*,i}(\mu_t^i - r_t) \right| dt &= \int_0^T \left| \sum_{i=1}^d \eta_t^i \, S_t^{*,i} \, \sigma_t^i \cdot \theta_t \right| dt \\ &\leq \left(\int_0^T \left| \sum_{i=1}^d \eta_t^i \, S_t^{*,i} \, \sigma_t^i \right|^2 dt \right)^{\frac{1}{2}} \left(\int_0^T |\theta_t|^2 \, dt \right)^{\frac{1}{2}} \\ &< \infty,\end{aligned}$$

by (5.43), (5.41), and condition (ii) of Theorem 5.4.1 satisfied by θ. On substituting the expression (5.36) for $\tilde{W}$ in (5.42) and using (5.33), it follows, since the integrals are all well defined, that the last line in (5.42) is equal P^*-a.s., and hence P-a.s., for all $t \in [0, T]$ to

$$M_0 + \int_0^t \left(\sum_{i=1}^d \eta_s^i \, S_s^{*,i}(\mu_s^i - r_s) \right) ds + \int_0^t \left(\sum_{i=1}^d \eta_s^i \, S_s^{*,i} \, \sigma_s^i \right) \cdot dW_s. \quad (5.44)$$

Here the last integral is a stochastic integral with respect to W that defines a continuous local martingale under P. It then follows from Lemma 5.2.2 that $\hat{\phi} = \eta$ can be extended, by the addition of another component, to a self-financing trading strategy ϕ with discounted value process $V^*(\phi)$ satisfying P-a.s.,

$$V_t^*(\phi) = M_t \quad \text{for all} \quad t \in [0, T]. \quad (5.45)$$

In particular, since P is equivalent to P^*, we have P^*-a.s.,

$$V_T^*(\phi) = M_T = X^*, \quad (5.46)$$

and hence $V_T(\phi) = X$ P^*-a.s. Since (5.45) holds and M is a martingale under P^*, it follows that ϕ is a replicating strategy for X.

(ii) implies (i): Assume that condition (ii) in the theorem holds and that Q is some equivalent local martingale measure for the model. We need to show that $Q = P^*$. Fix an event $A \in \mathcal{F}_T$. Let $X = 1_A S_T^0$, which is an $\mathcal{F}_T$-measurable random variable and satisfies $E^{P^*}[|X^*|] = E^{P^*}[1_A] \leq 1 < \infty$. By our hypothesis (ii), X is replicable (relative to P^*). So there exists an admissible strategy ϕ (relative to P^*) such that $V_T(\phi) = X$, P^*-a.s. By the admissibility of ϕ, the process $\{V_t^*(\phi), t \in [0, T]\}$ is a martingale under P^*.

By the replicating property of ϕ and the martingale property of $V^*(\phi)$ under P^*, we have for each $t \in [0, T]$, P^*-a.s.,

$$V_t^*(\phi) = E^{P^*}\left[V_T^*(\phi) \mid \mathcal{F}_t\right] = E^{P^*}\left[1_A \mid \mathcal{F}_t\right] \geq 0.$$

However, $V^*(\phi)$ has continuous paths, and so we can interchange the order of the qualifiers $t \in [0,T]$ and P^*-a.s. to conclude that P^*-a.s.,

$$V_t^*(\phi) \geq 0 \quad \text{for all} \quad t \in [0,T]. \tag{5.47}$$

Since P^* is equivalent to Q (as they are both equivalent to P), it follows that (5.47) also holds Q-a.s.

Using the replicating property of ϕ and the martingale property of $V^*(\phi)$ under P^* again, we have

$$\begin{aligned} P^*(A) &= E^{P^*}[1_A] = E^{P^*}[V_T^*(\phi)] \\ &= E^{P^*}[V_0^*(\phi)] = V_0(\phi). \end{aligned} \tag{5.48}$$

Now, by Corollary 5.4.4, since Q is an ELMM, $V^*(\phi)$ is a continuous *local* martingale under Q, and so there is a localizing sequence of stopping times $\{\tau_k\}$ such that $\tau_k \leq \tau_{k+1}$ for all k, $\tau_k \to \infty$ Q-a.s. as $k \to \infty$, and $\{V_{t\wedge\tau_k}^*(\phi), t \in [0,T]\}$ is a *martingale* under Q for each k. In particular, $V_T^*(\phi) = \lim_{k\to\infty} V_{T\wedge\tau_k}^*(\phi)$ Q-a.s. Since (5.47) holds Q-a.s., we have for each k, $V_{T\wedge\tau_k}^*(\phi) \geq 0$ Q-a.s. Thus, we may apply Fatou's lemma to conclude that

$$\begin{aligned} Q(A) &= E^Q[V_T^*(\phi)] \\ &= E^Q\left[\lim_{k\to\infty} V_{T\wedge\tau_k}^*(\phi)\right] \\ &\leq \liminf_{k\to\infty} E^Q[V_{T\wedge\tau_k}^*(\phi)] \\ &= \liminf_{k\to\infty} E^Q[V_0^*(\phi)] \\ &= V_0(\phi) \\ &= P^*(A). \end{aligned}$$

Here we have used Fatou's lemma for the inequality, the martingale property of the stopped process for the fourth line, and (5.48) for the last equality.

Since $A \in \mathcal{F}_T$ was arbitrary, it follows that $P^*(A) \geq Q(A)$ for *every* $A \in \mathcal{F}_T$. To complete the proof that (ii) implies (i) by contradiction, suppose we have the *strict* inequality $P^*(\tilde{A}) > Q(\tilde{A})$ for some $\tilde{A} \in \mathcal{F}_T$. Then, applying this inequality and the previous one (with $A = \tilde{A}^c = \Omega \setminus \tilde{A}$), we obtain

$$1 = P^*(\Omega) = P^*(\tilde{A}) + P^*(\tilde{A}^c) > Q(\tilde{A}) + Q(\tilde{A}^c) = 1.$$

This contradiction shows that $P^*(A) = Q(A)$ for all $A \in \mathcal{F}_T$. Hence $Q = P^*$, and we have established (i). $\square$

If condition (ii) of Theorem 5.5.1 is satisfied, then we say that the multi-dimensional Black-Scholes model is *complete*. By that same theorem, under our assumption that the model is viable, completeness is equivalent to uniqueness of the equivalent local martingale measure P^*.

To further characterize complete markets in the context of the multi-dimensional Black-Scholes model, we shall assume that the number of stocks d is not greater than the dimension n of the underlying Brownian motion W. One may argue that if this assumption does not hold, then the market model has a certain redundancy that can be eliminated by replacing the more than n stocks by n mutual funds that are time and realization dependent linear combinations of the stocks. We refer the reader to Karatzas and Shreve [**28**], Remark 1.4.10, for an explanation of this rationale.

Theorem 5.5.2. *Suppose that there exists an equivalent local martingale measure P^* and that $d \leq n$. Then the following two conditions are equivalent.*

(i) *The multi-dimensional Black-Scholes model is complete.*

(ii) *We have $d = n$, and P-a.s., the $d \times d$ matrix σ_t is invertible for a.e. $t \in [0, T]$.*

Remark. Note that since P is equivalent to P^*, in condition (ii) one could just as well use P^*-a.s. in place of P-a.s.

Proof. We give a nearly complete proof of this result. The main omission is in the proof that (i) implies (ii), where a construction of a suitable Borel measurable function ψ is assumed (we provide a reference for this result).

(ii) implies (i): Assume that $d = n$ and that P-a.s., σ_t is invertible for a.e. $t \in [0, T]$. Let Q be any equivalent local martingale measure for the model. Then by Theorem 5.4.3, there is a market price of risk $\theta = \{\theta_t, t \in [0, T]\}$ such that $\frac{dQ}{dP} = \Lambda_T$ where Λ_T is given by (5.20), and θ satisfies (5.33). Using the invertibility assumption on σ, it follows that P-a.s.,

$$\theta_t = \sigma_t^{-1}(\mu_t - r_t \mathbf{1}) \quad \text{for a.e. } t \in [0, T], \tag{5.49}$$

where σ_t^{-1} denotes the inverse of σ_t when it exists. This relation uniquely determines (up to almost sure equivalence) the integrals used in defining Λ_T, and hence Λ_T is uniquely determined by the model parameters σ, μ, and r. Since Λ_T determines Q from P, it follows that the measure Q is uniquely determined by the model parameters and P. Consequently, there is only one equivalent local martingale measure Q, and so we must have $Q = P^*$. Thus, there is a unique equivalent local martingale measure, and by the second fundamental theorem of asset pricing (Theorem 5.5.1), the model is complete.

(i) implies (ii): Suppose that (i) holds. In particular, suppose that (ii) of Theorem 5.5.1 holds. Consider the space $\mathcal{M}$ of $d \times n$ matrices endowed with the norm $|A| = (\sum_{i=1}^{d} \sum_{j=1}^{n} |A_{ij}|^2)^{\frac{1}{2}}$ for $A \in \mathcal{M}$. Viewing an element, A, of this space as a linear function $A : \mathbb{R}^n \to \mathbb{R}^d$, let $\mathcal{N}(A)$ denote the null space of A; i.e., $\mathcal{N}(A) = \{x \in \mathbb{R}^n : Ax = 0\}$. It is well known from linear algebra that $\mathcal{N}(A) = \mathcal{R}(A')^{\perp}$, where the latter denotes the orthogonal complement in $\mathbb{R}^n$ of the range $\mathcal{R}(A')$ of the (linear) transpose $A' : \mathbb{R}^d \to \mathbb{R}^n$ of A. One can show (see [**28**], Lemma 1.6.9 for a proof) that there is a bounded, Borel measurable function $\psi : \mathcal{M} \to \mathbb{R}^n$ such that for each $A \in \mathcal{M}$, $\psi(A) \in \mathcal{N}(A)$ and $\psi(A) \neq 0$ if $\mathcal{N}(A) \neq \{0\}$. Then the n-dimensional process κ, defined by the composition $\kappa_t = \psi(\sigma_t)$, $t \in [0, T]$, is bounded and adapted, $\kappa : [0, T] \times \Omega \to \mathbb{R}^n$ is $(\mathcal{B}_T \times \mathcal{F}_T)$-measurable where $\kappa(t, \omega) = \kappa_t(\omega)$, and for each $t \in [0, T]$, $\omega \in \Omega$, $\kappa_t(\omega) \in \mathcal{N}(\sigma_t(\omega))$ and $\kappa_t(\omega) \neq \{0\}$ if $\mathcal{N}(\sigma_t(\omega)) \neq \{0\}$. Let a market price of risk θ and process $\tilde{W}$ associated with P^* be as described in Corollary 5.4.4 with P^* in place of Q there. Since κ is bounded, the stochastic integral process

$$\int_0^t \kappa_s \cdot d\tilde{W}_s, \quad t \in [0, T],$$

defines a continuous L^2-martingale under P^*. Let

$$X = S_T^0 \left(1 + \int_0^T \kappa_t \cdot d\tilde{W}_t\right).$$

Then, X is $\mathcal{F}_T$-measurable and

$$E^{P^*}[|X^*|] \leq E^{P^*}\left[1 + \left|\int_0^T \kappa_t \cdot d\tilde{W}_t\right|\right] < \infty.$$

We also have by the martingale property of the stochastic integral that

$$E^{P^*}[X^*] = 1.$$

Since we are assuming that the model is complete, there is an admissible strategy (under P^*) that replicates X. We denote this strategy by ϕ. Then, $\{V_t^*(\phi), t \in [0, T]\}$ is a martingale under P^* and $V_T^*(\phi) = X^*$ P^*-a.s. Consequently,

$$V_0(\phi) = V_0^*(\phi) = E^{P^*}[V_T^*(\phi)] = E^{P^*}[X^*] = 1.$$

Since ϕ is self-financing, we also have by Corollary 5.4.4 that P^*-a.s. for all $t \in [0, T]$,

$$V_t^*(\phi) = V_0(\phi) + \int_0^t \left(\sum_{i=1}^{d} \phi_s^i \, S_s^{*,i} \, \sigma_s^i\right) \cdot d\tilde{W}_s.$$

Equating the values of $V_T^*(\phi)$ we have P^*-a.s.,

$$\begin{aligned} X^* &= 1 + \int_0^T \kappa_t \cdot d\tilde{W}_t \\ &= V_T^*(\phi) = 1 + \int_0^T \left(\sum_{i=1}^d \phi_t^i \, S_t^{*,i} \, \sigma_t^i \right) \cdot d\tilde{W}_t. \end{aligned}$$

Hence, upon subtracting we obtain, P^*-a.s.,

$$\int_0^T \left(\kappa_t - \sum_{i=1}^d \phi_t^i \, S_t^{*,i} \, \sigma_t^i \right) \cdot d\tilde{W}_t = 0.$$

Taking the conditional expectation $E^{P^*}[\,\cdot\,|\mathcal{F}_t]$ of both sides and using the martingale property of the stochastic integral under P^*, we obtain P^*-a.s., for all $t \in [0, T]$,

$$\int_0^t \left(\kappa_s - \sum_{i=1}^d \phi_s^i \, S_s^{*,i} \, \sigma_s^i \right) \cdot d\tilde{W}_s = 0.$$

The left member above defines a continuous martingale under P^*. Since the right member is zero, the quadratic variation process of this martingale must equal the zero process and so, P^*-a.s., for all $t \in [0, T]$,

$$\int_0^t \left| \kappa_s - \sum_{i=1}^d \phi_s^i \, S_s^{*,i} \, \sigma_s^i \right|^2 ds = 0.$$

It follows from this that P^*-a.s.,

$$\kappa_t = \sum_{i=1}^d \phi_t^i \, S_t^{*,i} \, \sigma_t^i = \sigma_t' \eta_t, \quad \text{for a.e.} \quad t \in [0, T],$$

where $\eta_t^i = \phi_t^i \, S_t^{*,i}$ for $i = 1, \ldots, d$, $t \in [0, T]$. Thus, P^*-a.s., $\kappa_t \in \mathcal{R}(\sigma_t')$ for a.e. $t \in [0, T]$. But by construction, $\kappa_t \in \mathcal{N}(\sigma_t) = \mathcal{R}(\sigma_t')^\perp$, and so it follows that P^*-a.s., $\kappa_t = 0$ for a.e. $t \in [0, T]$. Since $\kappa_t \neq 0$ whenever $\mathcal{N}(\sigma_t) \neq \{0\}$, it follows that P^*-a.s., $\mathcal{N}(\sigma_t) = \{0\}$ for a.e. $t \in [0, T]$. Hence, since we assumed that $d \leq n$, we must have that $d = n$ and that P^*-a.s., σ_t is invertible for a.e. $t \in [0, T]$. As P^* is equivalent to P, the conclusion that (ii) holds follows. □

5.6. Pricing European Contingent Claims

In this section, we assume that the multi-dimensional Black-Scholes model, which we view as the primary market model, is viable and complete. In this case there is a unique equivalent local martingale measure which we denote by P^*. Let θ and $\tilde{W}$ be as defined in Corollary 5.4.4 with P^* in place of Q there.

Consider a European contingent claim represented by $X \in \mathcal{F}_T$ and satisfying $E^{P^*}[|X^*|] < \infty$. We shall determine the unique arbitrage free price process for this claim. Our treatment is similar to that in Section 4.5. As usual, we need a notion of admissible strategy for trading in the stocks, money market and contingent claim.

We assume that any given price process $C = \{C_t, t \in [0,T]\}$ for the European contingent claim is a right continuous, adapted process, where C_t represents the price of the European contingent claim at time t. We suppose that for a given price process C, there is a set of admissible strategies Ψ_C for trading in the stocks, money market and contingent claim. At a minimum, an element ψ of Ψ_C should be a $(d+2)$-dimensional adapted process $\psi = \{\psi_t = (\phi_t, \gamma_t), t \in [0,T]\}$ where $\phi = \{\phi_t, t \in [0,T]\}$ is a $(d+1)$-dimensional adapted process representing the holdings over the time interval $[0,T]$ in the primary market assets (the money market instrument and the stocks), and $\gamma = \{\gamma_t, t \in [0,T]\}$ is a one-dimensional adapted process representing the holdings over $[0,T]$ in the contingent claim. We shall also want elements of Ψ_C to be self-financing in an appropriate way.

To show uniqueness of the arbitrage free price process, we do not want to place unnecessary restrictions on C just so that it can be used as an integrator (in defining self-financing trading strategies). In fact, to show uniqueness, we shall need only the reasonable assumption that rather simple strategies of the following type are admissible: those where one buys or sells one contingent claim, or does nothing, at some deterministic time t^*, and then holds that position in the claim until time T, but where one may trade as usual in the stocks and money market over $[t^*, T]$. More precisely, we assume that strategies of the following form are in Ψ_C:

$$\psi_t(\omega) = 1_A(\omega)(\tilde{\phi}_t^0(\omega) + \tilde{\delta}_t(\omega), \tilde{\phi}_t^1(\omega), \ldots, \tilde{\phi}_t^d(\omega), \tilde{\gamma}_t(\omega)), \quad t \in [t^*, T], \tag{5.50}$$

and $\psi_t(\omega) = 0$ for $t \in [0, t^*)$, for all $\omega \in \Omega$, where t^* is a fixed time in $[0,T]$, $A \in \mathcal{F}_{t^*}$, $\tilde{\phi} = \{(\tilde{\phi}_t^0, \tilde{\phi}_t^1, \ldots, \tilde{\phi}_t^d), t \in [0,T]\}$ is an admissible self-financing trading strategy in the primary (money market and stocks) market, $\tilde{\gamma}_t = +1$ for all $t \in [t^*, T]$ or $\tilde{\gamma}_t = -1$ for all $t \in [t^*, T]$, and $\tilde{\delta}_t = \tilde{\delta}_{t^*}$ for all $t \in [t^*, T]$ is chosen such that $\tilde{\phi}_{t^*} \cdot S_{t^*} + \tilde{\delta}_{t^*} S_{t^*}^0 + \tilde{\gamma}_{t^*} C_{t^*} = 0$. Under such a strategy, the investor does nothing prior to time t^*. On A^c, the investor continues to do nothing for the remaining interval $[t^*, T]$. On A, if $\tilde{\gamma}_{t^*} \equiv +1$, at t^* the investor buys one contingent claim for C_{t^*} and invests $-C_{t^*}$ in the money market instrument and stocks according to $(\tilde{\phi}_t^0 + \tilde{\delta}_t, \tilde{\phi}_t^1, \ldots, \tilde{\phi}_t^d)$ for $t \in [t^*, T]$. That is, on A, for $t \in [t^*, T]$, the investor follows the strategy $\tilde{\phi}$ in the primary market, and, in addition, purchases $\tilde{\delta}_{t^*}$ shares of the money market instrument at time t^* and holds those until time T. On A, if $\tilde{\gamma}_{t^*} \equiv -1$, instead of buying one contingent claim at time t^*, the investor sells one contingent claim at t^* and invests the proceeds C_{t^*} according to $(\tilde{\phi}_t^0 +$

$\tilde{\delta}_t, \tilde{\phi}_t^1, \ldots, \tilde{\phi}_t^d)$ for $t \in [t^*, T]$. Since the value of the strategy ψ is zero at time t^* and $\tilde{\delta}_t$ and $\tilde{\gamma}_t$ do not vary with $t \in [t^*, T]$, the strategy ψ will be naturally self-financing, since $\tilde{\phi}$ is assumed to have this property.

To show existence of the arbitrage free price process, we will prove below that when the price process C is a continuous modification of

$$S_t^0 E^{P^*}[X^* \mid \mathcal{F}_t], \quad t \in [0, T], \tag{5.51}$$

it is an arbitrage free price process. (A priori, one might expect that C should only be right continuous. However, it turns out that any right continuous modification of the above process is P^*-a.s. continuous as it is indistinguishable from the value of a replicating strategy for the European contingent claim.) For the existence proof, we need constraints on the admissible strategies similar to what we had in the case of the primary market. Note that if ϕ^* is a replicating strategy for X, then by the martingale property of $V^*(\phi^*)$ under P^*, we have $V_t^*(\phi^*) = E^{P^*}[X^* | \mathcal{F}_t]$ P^*-a.s. for each $t \in [0, T]$. So $C_t \equiv S_t^0 V_t^*(\phi^*)$, $t \in [0, T]$, is a continuous modification of (5.51). By Corollary 5.4.4 we have P^*-a.s.,

$$V_t^*(\phi^*) = V_0(\phi) + \int_0^t \left(\sum_{i=1}^d \phi_s^{*,i} \, S_s^{*,i} \, \sigma_s^i \right) \cdot d\tilde{W}_s, \quad t \in [0, T]. \tag{5.52}$$

Thus, given

$$C_t^* = C_t / S_t^0 = V_t^*(\phi^*), \ t \in [0, T], \tag{5.53}$$

it is natural to let the class of admissible strategies Ψ_C be the set of $(d+2)$-dimensional processes $\psi = (\phi, \gamma)$ such that

(i) $\psi : [0, T] \times \Omega \to \mathbb{R}^{d+2}$ is $(\mathcal{B}_T \times \mathcal{F}_T)$-measurable, where $\psi(t, \omega) = \psi_t(\omega)$ for all $t \in [0, T]$ and $\omega \in \Omega$;

(ii) ψ is adapted, i.e., $\psi_t \in \mathcal{F}_t$ for each $t \in [0, T]$;

(iii) ϕ is a $(d+1)$-dimensional process that is a trading strategy in the primary market consisting of the money market instrument and stocks;

(iv) γ is a one-dimensional process satisfying P^*-a.s.,

$$\int_0^T \gamma_s^2 \left| \sum_{i=1}^d \phi_s^{*,i} \, S_s^{*,i} \, \sigma_s^i \right|^2 ds < \infty; \tag{5.54}$$

(v) the value process $\{V_t(\psi), t \in [0, T]\}$ defined by

$$V_t(\psi) \equiv \phi_t \cdot S_t + \gamma_t C_t, \quad t \in [0, T], \tag{5.55}$$

is a continuous, adapted process such that the discounted value process $\{V_t^*(\psi), t \in [0, T]\}$ defined by

$$V_t^*(\psi) = V_t(\psi) / S_t^0, \quad t \in [0, T], \tag{5.56}$$

is a continuous martingale under P^* that satisfies P^*-a.s.,

$$\begin{aligned} V_t^*(\psi) &= V_0(\psi) + \int_0^t \left(\sum_{i=1}^d \phi_s^i \, S_s^{*,i} \, \sigma_s^i \right) \cdot d\tilde{W}_s \\ &\quad + \int_0^t \gamma_s \left(\sum_{i=1}^d \phi_s^{*,i} \, S_s^{*,i} \, \sigma_s^i \right) \cdot d\tilde{W}_s. \end{aligned} \tag{5.57}$$

Properties (i)–(iv) above ensure that the stochastic integrals appearing in (5.57), which is a form of self-financing condition on ψ, are well defined as continuous local martingales under P^*. Note that since S^* and C^* (given by (5.53)) are already martingales under P^*, the martingale condition on $V^*(\psi)$ amounts to some control on the size of ψ to rule out doubling strategies. One can verify that strategies of the form (5.50) satisfy the above properties (i)–(v) when C^* is given by (5.53).

Given a price process C for the contingent claim, an *arbitrage opportunity* in the stocks-money market-contingent claim market is an admissible strategy $\psi \in \Psi_C$ such that $V_0(\psi) = 0$, $V_T(\psi) \geq 0$ and $P^*(V_T(\psi) > 0) > 0$. (Since P and P^* are equivalent, we could just as easily have used P in place of P^* in the last inequality.)

In the following theorem, uniqueness means uniqueness up to indistinguishability of stochastic processes.

Theorem 5.6.1. *The unique arbitrage free price process for the European contingent claim X is a continuous modification of*

$$\{S_t^0 E^{P^*}[X^* \mid \mathcal{F}_t], \quad t \in [0,T]\}. \tag{5.58}$$

Proof. Consider a price process $\{C_t, t \in [0,T]\}$ for the European contingent claim. Since the primary market is assumed to be viable and complete, there is a replicating strategy ϕ^* for X, and its discounted value process $\{V_t^*(\phi^*), t \in [0,T]\}$ being a martingale under P^* with terminal value X^* must satisfy for each $t \in [0,T]$, P^*-a.s.,

$$V_t^*(\phi^*) = E^{P^*}[V_T^*(\phi^*) | \mathcal{F}_t] = E^{P^*}[X^* | \mathcal{F}_t].$$

Thus, the undiscounted value process $V(\phi^*)$, which is assumed to be continuous and adapted, is a continuous modification of $\{S_t^0 E^{P^*}[X^*|\mathcal{F}_t], t \in [0,T]\}$.

Suppose, for a proof of uniqueness by contradiction, that

$$P^*(C_t \neq V_t(\phi^*) \text{ for some } t \in [0,T]) > 0.$$

Since the process C is assumed to be right continuous and $V(\phi^*)$ is continuous, there exists a $t^* \in [0,T]$ such that $P^*(C_{t^*} \neq V_{t^*}(\phi^*)) > 0$. Let $A = \{\omega \in \Omega : C_{t^*}(\omega) > V_{t^*}(\phi^*)(\omega)\}$ and $\tilde{A} = \{\omega \in \Omega : C_{t^*}(\omega) < V_{t^*}(\phi^*)(\omega)\}$.

Then either $P^*(A) > 0$ or $P^*(\tilde{A}) > 0$. First suppose that $P^*(A) > 0$. Define $\psi = \{(\phi_t, \gamma_t), t \in [0,T]\}$ by

$$\psi_t(\omega) = \begin{cases} 0, & t < t^*, \\ 0, & t \geq t^*,\ \omega \in A^c, \\ \left(\phi_t^{*,0} + \frac{C_{t^*} - V_{t^*}(\phi^*)}{S^0_{t^*}}, \phi_t^{*,1}, \ldots, \phi_t^{*,d}, -1\right)(\omega), & t \geq t^*,\ \omega \in A. \end{cases}$$

The value of the portfolio represented by ψ_t is zero for $t \leq t^*$, and so the value at time zero is zero. Moreover, the value of ψ at time T is zero on A^c. Also, since ϕ^* is a replicating strategy for X, on A, the value of ψ at time T is $\left(\frac{C_{t^*} - V_{t^*}(\phi^*)}{S^0_{t^*}}\right) S^0_T$, which is strictly positive. Since $P^*(A) > 0$, ψ is an arbitrage opportunity.

Similarly, if $P^*(\tilde{A}) > 0$, then ψ defined for $t \in [0,T]$, $\omega \in \Omega$ by

$$\psi_t(\omega) = 1_{\{t \geq t^*\}} 1_{\tilde{A}}(\omega) \left(-\phi_t^{*,0} + \frac{V_{t^*}(\phi^*) - C_{t^*}}{S^0_{t^*}}, -\phi_t^{*,1}, \ldots, -\phi_t^{*,d}, 1\right)(\omega), \tag{5.59}$$

is an arbitrage opportunity.

Thus, we have shown that the only possible arbitrage free price process (up to indistinguishability) is given by $C_t = V_t(\phi^*)$, $t \in [0,T]$. Next we show that this price process is arbitrage free. For a contradiction, suppose that with this price process there is $\psi \in \Psi_C$ such that $V_0(\psi) = 0$, $V_T(\psi) \geq 0$ and $P^*(V_T(\psi) > 0) > 0$. Now, by the admissibility assumptions on ψ, the discounted value process $V^*(\psi)$ is a martingale under P^* and so

$$0 = V_0^*(\psi) = E^{P^*}[V_T^*(\psi)] = E^{P^*}\left[\frac{V_T(\psi)}{S^0_T}\right] > 0, \tag{5.60}$$

a contradiction. Thus, there cannot be an arbitrage opportunity with this price process. □

5.7. Incomplete Markets

Assuming the primary market model is viable, that is, there is at least one equivalent local martingale measure P^*, if the model is incomplete, then by the characterization afforded by the second fundamental theorem of asset pricing,

(i) there is more than one equivalent local martingale measure (ELMM) and

(ii) for any ELMM, P^*, there are European contingent claims whose discounted values are integrable under P^* but for which there is no replicating strategy.

Scrutiny of the proof of Theorem 5.6.1 reveals that if X is the value of a European contingent claim whose discounted value is integrable under some ELMM P^* and the European contingent claim is replicable relative to P^*, then a continuous modification of (5.58) is the unique arbitrage free price process for the European contingent claim. However, if X is not replicable relative to P^*, then, in a similar manner to that for finite market models, some additional criterion needs to be invoked to choose even an initial arbitrage free price for the European contingent claim. For further discussion of this, we refer the reader to Musiela and Rutkowski [**32**] and Karatzas and Shreve [**28**], for a start.

5.8. Exercises

In these exercises, we use the notation of the multi-dimensional Black-Scholes model.

1. The Black-Scholes model described in Chapter 4 can be thought of as a multi-dimensional Black-Scholes model with $d = n = 1$ and constant parameters μ, r, σ as described in Chapter 4. Show that in this case, the conditions placed on a trading strategy in Chapter 4 are equivalent to those described in this chapter.

2. Assume $d = n \geq 1$, μ, r, σ are constant processes and σ is invertible. Show that there is a unique equivalent local martingale measure and specify its form.

3. Using the results from Section 5.4, in particular, Corollary 5.4.4, prove the "only if" part of the first fundamental theorem of asset pricing. That is, prove that if there is an equivalent local martingale measure, then there is No Free Lunch with Vanishing Risk.

4. The following parameter choices for the multi-dimensional Black-Scholes model yield a simple stochastic volatility model. Assume $d = 1$, $n = 2$, β, r are constants, and the stock price process satisfies the equation

$$dS_t^1 = \beta S_t^1 \, dt + S_t^1 \, g(Z_t) \, dW_t^1, \tag{5.61}$$

where $g : (0, \infty) \to (0, \infty)$ is continuous and bounded below by a strictly positive constant, and Z is a continuous adapted process satisfying

$$dZ_t = \gamma Z_t \, dt + \delta Z_t \, dW_t^2, \tag{5.62}$$

γ, δ are constants, $\delta > 0$, and Z_0 is a strictly positive constant. In fact,

$$Z_t = Z_0 \exp\left(\gamma t + \delta W_t^2 - \frac{1}{2}\delta^2 t\right), \quad t \in [0, T],$$

is the unique continuous adapted process satisfying (5.62) given Z_0. Show that this model is viable and it is not complete.

Appendix A

Conditional Expectation and L^p-Spaces

In this appendix (A), we briefly review the definition and some basic properties of conditional expectation. We also establish some notation for L^p spaces. For more details we refer the reader to a graduate text in probability such as Chung [**9**] or D. Williams [**38**].

Consider a fixed probability space $(\Omega, \mathcal{F}, P)$, where Ω represents a sample space of possible outcomes, $\mathcal{F}$ is a σ-algebra of subsets of Ω representing the events to which probabilities can be assigned, and P is a probability measure on $(\Omega, \mathcal{F})$. Expectation under P will be denoted by $E[\cdot]$. The term a.s. will mean almost surely with respect to the probability measure P. Given a random variable X defined on this space satisfying $E[|X|] < \infty$ and a sub-σ-algebra $\mathcal{G} \subset \mathcal{F}$, the *conditional expectation* of X given $\mathcal{G}$ is denoted by $E[X|\mathcal{G}]$. This represents a member of the equivalence class of $\mathcal{G}$-measurable random variables satisfying

$$E[E[X|\mathcal{G}]1_A] = E[X1_A] \quad \text{for all } A \in \mathcal{G}. \tag{A.1}$$

Because conditional expectations are determined uniquely only up to almost sure equivalence, (in)equalities between conditional expectations are always interpreted to hold almost surely. If Y is a random variable defined on the probability space and $\sigma(Y)$ denotes the σ-algebra generated by Y, then the conditional expectation $E[X|\sigma(Y)]$ is often simply denoted by $E[X|Y]$. We list several common properties of conditional expectation here. In the

following, X and Y are integrable random variables defined on $(\Omega, \mathcal{F}, P)$, and $\mathcal{G}, \mathcal{H}$ are sub-σ-algebras of $\mathcal{F}$.

(i) If X is $\mathcal{G}$-measurable, then $E[X|\mathcal{G}] = X$.

(ii) If X is independent of $\mathcal{G}$, then $E[X|\mathcal{G}] = E[X]$.

(iii) If Z is $\mathcal{G}$-measurable and XZ is integrable (in addition to X), then

$$E[XZ|\mathcal{G}] = ZE[X|\mathcal{G}].$$

(iv) For $c \in \mathbb{R}$,

$$E[cX + Y|\mathcal{G}] = cE[X|\mathcal{G}] + E[Y|\mathcal{G}].$$

(v) If $X \leq Y$ a.s., then

$$E[X|\mathcal{G}] \leq E[Y|\mathcal{G}].$$

(vi) $|E[X|\mathcal{G}]| \leq E[|X||\mathcal{G}]$.

(vii) (Jensen's inequality) If $\phi : \mathbb{R} \to \mathbb{R}$ is convex and $\phi(X)$ is integrable (in addition to X), then

$$\phi(E[X|\mathcal{G}]) \leq E[\phi(X)|\mathcal{G}].$$

(viii) (Tower property) If $\mathcal{G} \subset \mathcal{H}$, then

$$E[E[X|\mathcal{H}]|\mathcal{G}] = E[E[X|\mathcal{G}]|\mathcal{H}] = E[X|\mathcal{G}].$$

(ix) (Monotone convergence) If, for each $n = 1, 2, \ldots$, X_n is a random variable such that $0 \leq X_n \leq X$ a.s., and $X_n \uparrow X$ a.s. as $n \to \infty$ (where X is integrable), then

$$E[X_n|\mathcal{G}] \uparrow E[X|\mathcal{G}] \quad \text{a.s. as } n \to \infty.$$

(x) (Dominated convergence) If, for each $n = 1, 2, \ldots$, X_n is a random variable such that $|X_n| \leq Y$ a.s., and $X_n \to X$ a.s. as $n \to \infty$ (where X, Y are integrable), then

$$E[X_n|\mathcal{G}] \to E[X|\mathcal{G}] \quad \text{a.s. as } n \to \infty.$$

The following special case is relevant to the probability models treated in Chapters 2 and 3. When the sample space Ω is a finite set, a sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$ can be specified by giving a finite partition $\mathcal{P} = \{A_1, \ldots, A_n\}$ of Ω that generates $\mathcal{G}$; i.e., A_i, $i = 1, \ldots, n$, are disjoint sets in $\mathcal{F}$, $\cup_{i=1}^n A_i = \Omega$ and $\mathcal{G} = \sigma(\mathcal{P})$, the smallest σ-algebra containing $\mathcal{P}$. In this case the random variable $Z = E[X|\mathcal{G}]$ must be constant on each of the sets in the partition $\mathcal{P}$, and its value on any set A_i of positive probability is given by

$$E[X1_{A_i}]/P(A_i).$$

For any σ-algebra $\mathcal{G} \subset \mathcal{F}$ and $p \in [1, \infty)$, $L^p(\Omega, \mathcal{G}, P)$ will denote the space of random variables $X : \Omega \to \mathbb{R}$ that are $\mathcal{G}$-measurable; i.e., $X^{-1}(B) \equiv$

$\{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{G}$ for each Borel set B in $\mathbb{R}$, and that satisfy $E[|X|^p] < \infty$. The norm of $X \in L^p(\Omega, \mathcal{G}, P)$ is given by

$$\|X\|_p = E[|X|^p]^{1/p}.$$

Furthermore, $L^\infty(\Omega, \mathcal{G}, P)$ denotes the space of $\mathcal{G}$-measurable random variables $X : \Omega \to \mathbb{R}$ such that X is bounded P-a.s. The norm on this space is the so-called essential supremum norm given for $X \in L^\infty(\Omega, \mathcal{G}, P)$ by

$$\|X\|_\infty = \inf\{a \geq 0 : P(|X| > a) = 0\}.$$

The following interpretation of $E[X|\mathcal{G}]$ is useful when X is square integrable. If the random variable X is such that $E[X^2] < \infty$, then the conditional expectation $Z = E[X|\mathcal{G}]$ is a version of the orthogonal projection of X onto the L^2 space $L^2(\Omega, \mathcal{G}, P)$; i.e., Z is the $\mathcal{G}$-measurable, square integrable random variable (unique up to a.s. equivalence) that minimizes the squared distance

$$E[(Z - X)^2].$$

Appendix B

Discrete Time Stochastic Processes

In this appendix (B), we briefly review some concepts and results for the types of discrete time stochastic processes used in this book. For more details and further background we refer the reader to a graduate text in probability such as Chung [**9**] or D. Williams [**38**].

Only discrete time stochastic processes indexed by a finite set of times are considered in this appendix, and so, without loss of generality, here we assume that the time index set is the set of consecutive non-negative integers $\{0, 1, \ldots, T\}$ for some finite non-negative integer T.

Throughout this appendix (B), we assume that $(\Omega, \mathcal{F}, P)$ is a fixed probability space, where Ω is a sample space representing the set of possible outcomes, $\mathcal{F}$ is a σ-algebra of subsets of Ω representing the events to which we can assign probabilities, and P is a probability measure on $(\Omega, \mathcal{F})$. The expectation with respect to P will be denoted by $E[\cdot]$.

A *filtration* is a family $\{\mathcal{F}_t, t = 0, 1, \ldots, T\}$ of sub-σ-algebras of $\mathcal{F}$ indexed by $t = 0, 1, \ldots, T$ such that

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \ldots \subset \mathcal{F}_T;$$

i.e., the family is increasing with time. If the sample space Ω is a finite set, often the σ-algebra $\mathcal{F}_0$ is trivial, consisting simply of the empty set $\emptyset$ and the whole sample space Ω. Also, since we shall be considering only random variables that are $\mathcal{F}_T$-measurable, without loss of generality (by redefining $\mathcal{F}$ to equal $\mathcal{F}_T$), we can assume that $\mathcal{F}_T = \mathcal{F}$. We often write $\{\mathcal{F}_t\}$ instead of the more cumbersome $\{\mathcal{F}_t, t = 0, 1, \ldots, T\}$. Intuitively, the filtration keeps track of what information is known at each of the times $t = 0, 1, \ldots, T$, where

information only increases with time. More precisely, for each $t = 0, 1, \dots, T$, the σ-algebra $\mathcal{F}_t$ tells us which events may be observed by time t. We call the quadruple $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ a *filtered probability space.*

For the remainder of this appendix (B) we suppose that in addition to $(\Omega, \mathcal{F}, P)$ being fixed, $\{\mathcal{F}_t\}$ is a fixed filtration, and so $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ is a given filtered probability space. All random variables and processes considered in this appendix are assumed to be defined on this space.

Given a positive integer d, a *d-dimensional (stochastic) process* with time index set $\{0, 1, \dots, T\}$, defined on the given filtered probability space, is a collection $X = \{X_t, t = 0, 1, \dots, T\}$ where each X_t is a d-dimensional random vector; i.e., a function $X_t : \Omega \to \mathbb{R}^d$ such that $X_t^{-1}(B) \equiv \{\omega \in \Omega : X_t(\omega) \in B\} \in \mathcal{F}$ for each Borel subset B of $\mathbb{R}^d$. The process X is called *adapted* if $X_t^{-1}(B) \in \mathcal{F}_t$ for each Borel set B in $\mathbb{R}^d$ and for each $t = 0, 1, \dots, T$. We often write $X_t \in \mathcal{F}_t$ as shorthand for $X_t^{-1}(B) \in \mathcal{F}_t$ for all Borel sets B in $\mathbb{R}^d$.

Given two d-dimensional processes $Y = \{Y_t,\ t = 0, 1, \dots, T\}$ and $Z = \{Z_t,\ t = 0, 1, \dots, T\}$ defined on $(\Omega, \mathcal{F}, P)$, these processes are *modifications of one another* if $P(Y_t = Z_t) = 1$ for each $t = 0, 1, \dots, T$. Since the time index set is finite, this is equivalent to Y and Z being *indistinguishable*; i.e., $P(Y_t = Z_t \text{ for } t = 0, 1, \dots, T) = 1$. We shall regard two such indistinguishable processes as being equal as stochastic processes.

A (discrete) *stopping time* (or optional time) is a function $\tau : \Omega \to \{0, 1, \dots, T\} \cup \{\infty\}$ such that

$$\{\tau = t\} \in \mathcal{F}_t \quad \text{for} \quad t = 0, 1, \dots, T. \tag{B.1}$$

Note that for such a stopping time τ, we have

$$\{\tau = \infty\} = \Omega \setminus (\cup_{t=0}^{T} \{\tau = t\}) \in \mathcal{F}_T.$$

For convenience we define $\mathcal{F}_\infty = \mathcal{F}_T$ and then (B.1) also holds with $t = \infty$. A deterministic time in $\{0, 1, \dots, T\} \cup \{\infty\}$ is a stopping time. Furthermore, if τ and σ are two stopping times, then it is straightforward to verify that $\tau \wedge \sigma = \min(\tau, \sigma)$ and $\tau \vee \sigma = \max(\tau, \sigma)$ are also stopping times. There is a σ-algebra associated with any (discrete) stopping time τ, defined by

$$\mathcal{F}_\tau = \{A \in \mathcal{F}_\infty : A \cap \{\tau = t\} \in \mathcal{F}_t \ \text{ for } \ t = 0, 1, \dots, T\}.$$

Note that, for $A \in \mathcal{F}_\tau$, $A \cap \{\tau = \infty\} \in \mathcal{F}_\infty$.

A collection $M = \{M_t, \mathcal{F}_t, t = 0, 1, \dots, T\}$, where each M_t is a real-valued random variable, is called a *martingale* if the following three properties hold:

(i) $E[|M_t|] < \infty$ for $t = 0, 1, \dots, T$,

(ii) M_t is $\mathcal{F}_t$-measurable for $t = 0, 1, \dots, T$,

(iii) $E[M_t|\mathcal{F}_{t-1}] = M_{t-1}$ for $t = 1, \ldots, T$.

In this discrete time setting, condition (iii) can be equivalently replaced by

(iii)$'$ $E[M_t|\mathcal{F}_s] = M_s$ for all $s < t$ in $\{0, 1, \ldots, T\}$.

We call M a *submartingale* if the "$=$" in (iii) (or (iii)$'$) is replaced by "$\geq$", and we call M a *supermartingale* if the "$=$" in (iii) (or (iii)$'$) is replaced by "$\leq$". The condition (iii) (or (iii)$'$) is often referred to as the (sub/super)martingale property. Note that M is a submartingale if and only if $-M$ is a supermartingale. If M is a (sub/super)martingale and in addition there is $p \in (1, \infty)$ such that $M_t \in L^p(\Omega, \mathcal{F}, P)$ for each $t = 0, 1, \ldots, T$, then we call M an L^p-(sub/super)martingale. In describing (sub/super)martingales, we shall sometimes omit the filtration $\{\mathcal{F}_t\}$ from the notation for M when it is understood.

We adopt the convention that for a d-dimensional process M, the collection $\{M_t, \mathcal{F}_t, t = 0, 1, \ldots, T\}$ is called a (sub/super)martingale if and only if each of its one-dimensional components $\{M_t^i, \mathcal{F}_t, t = 0, 1, \ldots, T\}$, $i = 1, \ldots, d$, is a (sub/super)martingale.

Two basic results from discrete time martingale theory are stated here for ease of reference. We state them for one-dimensional processes.

Theorem B.0.1. *(Doob's L^p-inequality) For $p \in (1, \infty)$, let $\{M_t, \mathcal{F}_t, t = 0, 1, \ldots, T\}$ be a real-valued L^p-martingale. Then, for each $t = 0, 1, \ldots, T$, $N_t \equiv \sup\{|M_s|, s = 0, 1, \ldots, t\} \in L^p(\Omega, \mathcal{F}_t, P)$ and*

$$E\left[(N_t)^p\right] \leq q^p E\left[|M_t|^p\right], \tag{B.2}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem B.0.2. *(Doob's stopping theorem) Let $\{M_t, \mathcal{F}_t, t = 0, 1, \ldots, T\}$ be a real-valued (sub)martingale and suppose that τ is a stopping time. Then $\{M_{t\wedge\tau}, \mathcal{F}_{t\wedge\tau}, t = 0, 1, \ldots, T\}$ is a (sub)martingale. Furthermore, if σ is another stopping time and $\sigma \leq \tau \leq T$, then M_τ, M_σ are integrable random variables and*

$$E[M_\tau] \geq E[M_\sigma], \tag{B.3}$$

where equality holds if M is a martingale.

We shall use the last result above in particular with $\sigma = 0$.

Appendix C

Continuous Time Stochastic Processes

In this appendix (C), we briefly review some concepts and results from the theory of continuous time stochastic processes. Many of these concepts and results are similar to those introduced for discrete time stochastic processes in the preceding appendix. Some notable exceptions are the definitions of stopping time and martingale and the fact that filtrations and stochastic processes in continuous time are frequently assumed to have some regularity (e.g., right continuity) due to the fact that the time index set is now a continuum. In this appendix (C), we restrict ourselves to compact time intervals.

In the following, we consider the compact time interval $[0, T]$ where T is a fixed value in $[0, \infty)$. We denote the Borel σ-algebra on $[0, T]$ by $\mathcal{B}_T$.

A *filtered probability space* is a quadruple $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \in [0, T]\}, P)$, where Ω is a set representing the sample space, $\mathcal{F}$ is a σ-algebra of subsets of Ω, P is a probability measure on $(\Omega, \mathcal{F})$, and $\{\mathcal{F}_t, t \in [0, T]\}$ is a filtration; i.e., a family of sub-σ-algebras of $\mathcal{F}$ indexed by $t \in [0, T]$ that is increasing: for each $s, t \in [0, T]$ satisfying $s < t$ we have $\mathcal{F}_s \subset \mathcal{F}_t$. We often write $\{\mathcal{F}_t\}$ instead of the more cumbersome $\{\mathcal{F}_t, t \in [0, T]\}$. Intuitively, we may regard $\mathcal{F}$ as containing all events which might ever be observed or to which we can assign probabilities. The filtration keeps track of what information is known by each of the times $t \in [0, T]$, where information only increases with time. More precisely, for each $t \in [0, T]$, the σ-algebra $\mathcal{F}_t$ tells us which events might be observed by time t. For such a filtered probability space, a set $A \in \mathcal{F}$ whose probability is zero is called a *P-null set.* The expectation with respect to P will be denoted by $E[\cdot]$. Since we shall be

considering only random variables that are $\mathcal{F}_T$-measurable, without loss of generality (by redefining $\mathcal{F}$ to equal $\mathcal{F}_T$), we can assume that $\mathcal{F}_T = \mathcal{F}$.

In continuous time, some additional regularity assumptions are typically imposed on a filtered probability space. In particular, a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ is said to satisfy the *usual conditions* if the following three conditions hold:

(i) $(\Omega, \mathcal{F}, P)$ is complete, i.e., if $A \subset B$ where $B \in \mathcal{F}$ and $P(B) = 0$, then $A \in \mathcal{F}$ and $P(A) = 0$;

(ii) $\mathcal{F}_0$ contains all of the P-null sets;

(iii) $\{\mathcal{F}_t\}$ is right continuous, i.e., $\mathcal{F}_t = \mathcal{F}_{t+} \equiv \bigcap_{s \in (t,T]} \mathcal{F}_s$, for all $t < T$.

Remark. If we have a filtered probability space that does not satisfy the usual conditions, by a standard procedure of completing the probability space, augmenting the members of the filtration using all of the P-null sets, and replacing $\mathcal{F}_t$ by $\mathcal{F}_{t+}$ for $t < T$, we can ensure that the usual conditions are satisfied (cf. Chung [**9**], page 29ff.).

For the remainder of this appendix, we suppose that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ is a given filtered probability space satisfying the usual conditions. All random variables and processes considered in this section are assumed to be defined on this space.

Given a positive integer d, a *d-dimensional (stochastic) process* on the given filtered probability space is a collection $X = \{X_t, t \in [0,T]\}$ where each X_t is a d-dimensional random vector; i.e., a function $X_t : \Omega \to \mathbb{R}^d$ such that $X_t^{-1}(B) \equiv \{\omega \in \Omega : X_t(\omega) \in B\} \in \mathcal{F}$ for each Borel subset B of $\mathbb{R}^d$. (In fact, it is enough to check that this last property holds for all open balls B in $\mathbb{R}^d$.) The process X is called *adapted* if $X_t^{-1}(B) \in \mathcal{F}_t$ for each Borel set B in $\mathbb{R}^d$ and for all $t \in [0,T]$. We often write $X_t \in \mathcal{F}_t$ as shorthand for $X_t^{-1}(B) \in \mathcal{F}_t$ for all Borel sets B in $\mathbb{R}^d$. By setting $X(t,\omega) = X_t(\omega)$ for all $t \in [0,T]$, $\omega \in \Omega$, we may sometimes view X as a function from $[0,T] \times \Omega$ into $\mathbb{R}^d$. The process X is said to be (left/right) continuous if all of its sample paths are (left/right) continuous. (Some authors require only that (left/right) continuous processes have (left/right) continuous paths almost surely. Frequently a process with such a property can be redefined on a P-null set to have (left/right) continuous paths surely. For most practical purposes such a redefined process can be considered to be the same as the original process.)

Two d-dimensional processes $\{Y_t, t \in [0,T]\}$ and $\{Z_t, t \in [0,T]\}$ defined on $(\Omega, \mathcal{F}, P)$ are

(i) *modifications of one another* if $P(Y_t = Z_t) = 1$ for all $t \in [0,T]$,

(ii) *indistinguishable* if $P(Y_t = Z_t \text{ for all } t \in [0,T]) = 1$.

(Note that for (ii) it is implicit that the event $\{Y_t = Z_t \text{ for all } t \in [0,T]\}$ is measurable.) Two right continuous processes that are modifications of one another are indistinguishable, and we shall regard them as being equal.

A *stopping time* (or optional time) is a function $\tau : \Omega \to [0,T] \cup \{\infty\}$ such that

$$\{\tau \le t\} \in \mathcal{F}_t \quad \text{for all} \quad t \in [0,T]. \tag{C.1}$$

Note that for such a stopping time τ, we have $\{\tau = \infty\} = \Omega \backslash \{\tau \le T\} \in \mathcal{F}_T$, and so we define $\mathcal{F}_\infty = \mathcal{F}_T$. Note that then (C.1) also holds with $t = \infty$. There is a σ-algebra associated with any stopping time τ, defined by

$$\mathcal{F}_\tau = \{A \in \mathcal{F}_\infty : A \cap \{\tau \le t\} \in \mathcal{F}_t \;\text{ for all }\; t \in [0,T]\}\}.$$

It is easily verified that a stopping time τ is measurable as a mapping from $(\Omega, \mathcal{F}_\tau)$ into $([0,T] \cup \{\infty\}, \mathcal{B}_\infty)$ where $\mathcal{B}_\infty$ is the σ-algebra on $[0,T] \cup \{\infty\}$ generated by the Borel sets in $[0,T]$ together with the singleton set $\{\infty\}$. If τ and σ are two stopping times, then it is straightforward to verify that $\tau \wedge \sigma = \min(\tau, \sigma)$ and $\tau \vee \sigma = \max(\tau, \sigma)$ are also stopping times.

Example. If X is a continuous, adapted d-dimensional process, then for any open or closed set $A \subset \mathbb{R}^d$, $\tau_A = \inf\{t \in [0,T] : X_t \in A\}$ is a stopping time. (As usual, the infimum of the empty set is defined to equal ∞.) For a proof of this that uses the fact that the filtration satisfies the usual conditions, see Chung [**9**], Section 2.4.

A collection $M = \{M_t, \mathcal{F}_t, t \in [0,T]\}$, where each M_t is a real-valued random variable, is called a *martingale* if the following three properties hold:

(i) $E[|M_t|] < \infty$ for all $t \in [0,T]$,

(ii) M_t is $\mathcal{F}_t$-measurable for each $t \in [0,T]$,

(iii) $E[M_t | \mathcal{F}_s] = M_s$ for all $s \le t$ in $[0,T]$.

We call M a *submartingale* if the "=" in (iii) is replaced by "$\ge$", and we call M a *supermartingale* if the "=" in (iii) is replaced by "$\le$". This condition (iii) is referred to as the (sub/super)martingale property whenever M is a (sub/super)martingale. If M is a (sub/super)martingale and in addition there is $p \in (1,\infty)$ such that $M_t \in L^p(\Omega, \mathcal{F}, P)$ for all $t \in [0,T]$, then we call M an L^p-(sub/super)martingale.

It is well known that under the usual conditions that we have assumed to hold for the filtered probability space, every martingale has a modification whose paths are all right continuous with finite left limits; cf. Chung [**9**], Theorem 3, page 29, and Corollary 1, page 26.

The stochastic integrals that we consider will often define martingales; however, sometimes they will only locally define martingales. This leads us to the notion of a local martingale. A collection $M = \{M_t, \mathcal{F}_t, t \in [0,T]\}$, where each M_t is a real-valued random variable, is called a *local martingale*

if M is an adapted process and there is a sequence of stopping times $\{\tau_k\}_{k=1}^{\infty}$ such that $\tau_k \leq \tau_{k+1}$ for each k, $\lim_{k\to\infty} \tau_k = \infty$ P-a.s., and for each k,

$$M^k = \{M_{t\wedge\tau_k}, \mathcal{F}_t, t \in [0,T]\}$$

is a martingale. We call such a sequence $\{\tau_k\}$ a localizing sequence for M. Here we have used a slightly stronger notion of local martingale than that introduced in [**11**] and [**35**]. In the case when the initial value M_0 is a constant, our definition is equivalent to that in [**11**] and [**35**]. As that is the only circumstance considered here, we have chosen to give the stronger and simpler definition. Indeed, for stochastic integrals, the initial value is not only constant but zero.

In describing (sub/super/local) martingales, we shall sometimes omit the filtration $\{\mathcal{F}_t\}$ from the notation for M when it is understood. We also adopt the convention that a d-dimensional process $M = \{M_t, t \in [0,T]\}$ is called a (sub/super/local) martingale if and only if each of its one-dimensional components $\{M_t^i, t \in [0,T]\}$, $i = 1, \ldots, d$, is a (sub/super/local) martingale. In the case of a local martingale, one can choose a common localizing sequence for all components, since the minimum of finitely many stopping times is again a stopping time. We shall always assume that such a common sequence is being used when we consider multi-dimensional local martingales.

Two basic results from continuous time martingale theory are stated here for ease of reference. We state them for one-dimensional processes.

Theorem C.0.3. *(Doob's L^p-inequality) For $p \in (1,\infty)$, let $\{M_t, \mathcal{F}_t, t \in [0,T]\}$ be a real-valued right continuous L^p-martingale. Then, for each $t \in [0,T]$, $N_t \equiv \sup_{0\leq s\leq t} |M_s| \in L^p(\Omega, \mathcal{F}_t, P)$ and*

$$E\left[(N_t)^p\right] \leq q^p E\left[|M_t|^p\right], \tag{C.2}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem C.0.4. *(Doob's Stopping Theorem) Let $\{M_t, \mathcal{F}_t, t \in [0,T]\}$ be a real-valued right continuous (sub)martingale and suppose that τ is a stopping time. Then $\{M_{t\wedge\tau}, \mathcal{F}_{t\wedge\tau}, t \in [0,T]\}$ is a right continuous (sub)martingale. Furthermore, if σ is another stopping time and $\sigma \leq \tau \leq T$, then M_τ and M_σ are integrable random variables and*

$$E[M_\tau] \geq E[M_\sigma], \tag{C.3}$$

where equality holds if M is a martingale.

We shall use the last result above in particular with $\sigma = 0$.

Appendix D

Brownian Motion and Stochastic Integration

In this appendix (D), we briefly review some definitions and basic results for Brownian motion and stochastic integrals with respect to Brownian motion. For more details and further background we refer the reader to Chung [**9**] and Chung and Williams [**11**].

All stochastic processes considered in this appendix are indexed by time lying in the compact time interval $[0,T]$ for a fixed value T in $[0,\infty)$, and all processes are assumed to be defined on a given complete probability space $(\Omega,\mathcal{F},P)$.

D.1. Brownian Motion

A standard *one-dimensional Brownian motion* (on the time interval $[0,T]$) is a one-dimensional process $W=\{W_t, t\in[0,T]\}$ such that

(i) $W_0 = 0$ P-a.s.;

(ii) all of the sample paths of W are continuous;

(iii) W has independent increments: $W_{t_1}-W_{t_0},\dots,W_{t_n}-W_{t_{n-1}}$ are independent for any $0\le t_0<t_1,<\cdots<t_n<\infty$, $n=1,2,\dots$;

(iv) for $0\le s<t<\infty$, $W_t - W_s$ is a normally distributed random variable with mean zero and variance $t-s$.

One can think of standard Brownian motion as a limit in distribution of a sequence of simple symmetric random walks where the time step and walk step size are scaled to zero in a suitable way (see e.g., [**27**]).

For a positive integer $n \geq 1$, a standard *n-dimensional Brownian motion* is an n-dimensional process $W = \{W_t, t \in [0,T]\}$ such that the coordinate processes $W^1, W^2, \ldots, W^n$ are mutually independent and for each $i \in \{1, \ldots, n\}$, $W^i = \{W_t^i, t \in [0,T]\}$ is a standard one-dimensional Brownian motion.

Assuming that W is a standard n-dimensional Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, P)$, for each $t \in [0,T]$, let $\mathcal{F}_t^o = \sigma\{W_s : 0 \leq s \leq t\}$, the smallest σ-algebra on Ω with respect to which W_s is measurable for each $0 \leq s \leq t$. For each $t \in [0,T]$, let $\mathcal{F}_t$ denote the smallest σ-algebra containing $\mathcal{F}_t^o$ and all of the P-null sets in $\mathcal{F}$. This is called the augmentation of $\mathcal{F}_t$ by the P-null sets, and $\{\mathcal{F}_t, t \in [0,T]\}$ is called the filtration generated by the Brownian motion W under P. The latter is sometimes referred to as the *standard filtration* associated with the Brownian motion or simply the Brownian filtration. It is well known that this filtration is right continuous; i.e., for each $t \in [0,T)$, $\mathcal{F}_t = \mathcal{F}_{t+} \equiv \cap_{s\in(t,T]}\mathcal{F}_s$ (cf. Chung [**9**], Section 2.3, Theorem 4). Hence the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ satisfies the usual conditions. It is well known that in this case when the filtration is the standard Brownian filtration, every martingale relative to $\{\mathcal{F}_t\}$ has a continuous modification; cf. Theorem V.3.5 in Revuz and Yor [**35**]. It is straightforward to check that $\{W_t, \mathcal{F}_t, t \in [0,T]\}$ is a continuous, n-dimensional martingale.

D.2. Stochastic Integrals (with respect to Brownian motion)

In Chapter 3 sums of the form:

$$\sum_{s=1}^{t} \phi_s \cdot \Delta S_s^*$$

play a key role in arbitrage pricing of European contingent claims in discrete finite market models. For the continuous market models considered in Chapters 4 and 5, we need the continuous time analogue of these discrete time "stochastic integrals" where S^* is replaced by a solution of a stochastic differential equation driven by Brownian motion. This leads us to define stochastic integrals where Brownian motion is the integrator. In general, these stochastic integrals cannot be defined pathwise using Riemann-Stieltjes integrals, since the paths of Brownian motion although continuous are not locally of bounded variation.

Suppose that W is a standard one-dimensional Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, P)$ and $\{\mathcal{F}_t\}$ is a filtration such that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ satisfies the usual conditions and $\{W_t, \mathcal{F}_t, t \in [0,T]\}$ is a continuous martingale. (One could use the standard filtration associated

with the Brownian motion for $\{\mathcal{F}_t\}$; however, we do not assume that a priori.) Without loss of generality, we assume that $\mathcal{F} = \mathcal{F}_T$.

Let $\mathcal{L}_s$ denote the set of one-dimensional processes $Y = \{Y_t, t \in [0,T]\}$ such that Y can be written in the simple form:

$$Y_t(\omega) = H_0(\omega)1_{\{0\}}(t) + \sum_{i=1}^{n-1} H_i(\omega)1_{(t_i,t_{i+1}]}(t) \quad \text{for all } t \in [0,T],\ \omega \in \Omega,$$

for some $0 = t_1 < t_2 < \cdots < t_n = T < \infty$, $H_i : \Omega \to \mathbb{R}$ that is bounded and $\mathcal{F}_{t_i}$-measurable for $i = 0, 1, \ldots, n-1$, where $t_0 \equiv 0$. Note that for each fixed ω as a function of t, Y is a step function. Also, the value of Y_0 depends only on the information available at time zero, and for each $i = 1, \ldots, n-1$, the value of the random variable Y_t for t in the interval $(t_i, t_{i+1}]$ depends only on the information available up to time t_i. For these reasons Y is called a *simple predictable* function. (The σ-algebra induced on $[0,T] \times \Omega$ by the simple predictable processes is called the *predictable* σ-algebra.)

Define for each $\omega \in \Omega$ and $t \in [0,T]$,

$$\left(\int_0^t Y_s\, dW_s\right)(\omega) = \sum_{i=1}^{n-1} H_i(\omega)(W_{t\wedge t_{i+1}} - W_{t\wedge t_i})(\omega).$$

This family of stochastic integrals (one integral for each time t) is linear in $Y \in \mathcal{L}_s$ and has the following properties:

(i) $\left\{\int_0^t Y_s\, dW_s, \mathcal{F}_t, t \in [0,T]\right\}$ is a continuous L^2-martingale,

(ii) $E\left[\left(\int_0^t Y_s\, dW_s\right)^2\right] = E\left[\int_0^t Y_s^2\, ds\right]$ for all $t \in [0,T]$,

(iii) $E\left[\sup_{t\in[0,T]} \left(\int_0^t Y_s\, dW_s\right)^2\right] \leq 4E\left[\int_0^T Y_s^2\, ds\right]$.

The first two properties follow by simple but tedious verification. Property (ii) is referred to as the L^2-isometry. The third property is a consequence of (i) and (ii) and Doob's inequality for continuous L^2-martingales.

Let $\mathcal{L}$ denote the set of all one-dimensional processes $Y = \{Y_t, t \in [0,T]\}$ such that Y is adapted, $Y : [0,T] \times \Omega \to \mathbb{R}$ is $(\mathcal{B}_T \times \mathcal{F}_T)$-measurable where $Y(t,\omega) = Y_t(\omega)$ for all $t \in [0,T]$ and $\omega \in \Omega$, and

$$E\left[\int_0^T Y_s^2\, ds\right] < \infty.$$

Then $\mathcal{L} \subset L^2([0,T] \times \Omega, \mathcal{B}_T \times \mathcal{F}_T, \lambda \times P)$, where λ denotes Lebesgue measure on $[0,T]$. The L^2-norm of an element $Y \in \mathcal{L}$ is given by

$$\|Y\|_{\mathcal{L}} = \left(E\left[\int_0^T Y_s^2\, ds\right]\right)^{\frac{1}{2}}.$$

It is known (cf. Chung and Williams [**11**], Sections 3.3 and 2.5) that $\mathcal{L}_s$ is dense in $\mathcal{L}$ with respect to the L^2-norm $\|\cdot\|_{\mathcal{L}}$. This property, the isometry (ii), and its consequence (iii) are the keys to extending the definition of the stochastic integral process $\{\int_0^t Y_s\,dW_s,\ t \in [0,T]\}$ to all integrands $Y \in \mathcal{L}$. Briefly, this extension is accomplished as follows. For any $Y \in \mathcal{L}$, there is a sequence $\{Y^{(m)}\}_{m=1}^{\infty}$ in $\mathcal{L}_s$ such that $\|Y - Y^{(m)}\|_{\mathcal{L}} \to 0$ as $m \to \infty$. It follows that such a sequence is Cauchy in $\mathcal{L}$ and hence by the linearity of the stochastic integral and the isometry property (ii) (with $t = T$), the corresponding sequence of stochastic integrals $\{\int_0^T Y_s^{(m)}\,dW_s\}_{m=1}^{\infty}$ is Cauchy in $L^2(\Omega, \mathcal{F}_T, P)$. In fact, using (iii) on differences of the stochastic integral processes and a standard Borel-Cantelli argument, one can show that there is a subsequence $\{m_k\}$ of $\{m\}$ and a continuous, adapted process, which we denote by $\{\int_0^t Y_s\,dW_s,\ t \in [0,T]\}$, such that P-a.s., as $k \to \infty$, $\int_0^t Y_s^{(m_k)}\,dW_s$ converges uniformly for $t \in [0,T]$ to $\int_0^t Y_s\,dW_s$. Up to indistinguishability, this limit process $\{\int_0^t Y_s\,dW_s,\ t \in [0,T]\}$ does not depend on the particular convergent subsequence nor on the original sequence chosen to approximate Y. This process $\{\int_0^t Y_s\,dW_s,\ t \in [0,T]\}$ inherits the martingale and isometry properties, (i) and (ii), from the approximating sequence.

A final step in defining stochastic integrals is to localize the class of integrands with the attendant stochastic integrals yielding local martingales starting from zero. For this, let $\mathcal{L}_{loc}$ denote the set of one-dimensional processes $Y = \{Y_t, t \in [0,T]\}$ such that Y is adapted, $Y : [0,T] \times \Omega \to \mathbb{R}$ is $(\mathcal{B}_T \times \mathcal{F}_T)$-measurable where $Y(t,\omega) = Y_t(\omega)$ for all $t \in [0,T]$ and $\omega \in \Omega$, and

$$\int_0^T Y_s^2\,ds < \infty \qquad P\text{-a.s.}$$

Given $Y \in \mathcal{L}_{loc}$, for each positive integer m, consider the stopping time

$$\tau_m = \inf\left\{t \in [0,T] : \int_0^t Y_s^2\,ds \geq m\right\}.$$

It is readily verified that for each m, the process $Y^{(m)}$ defined by

$$Y_s^{(m)}(\omega) = 1_{[0,\tau_m(\omega)]}(s) Y_s(\omega) \qquad \text{for } s \in [0,T],\ \omega \in \Omega,$$

is in $\mathcal{L}$. One can show that, as m increases, the stochastic integral processes $\{\int_0^t Y_s^{(m)}\,dW_s,\ t \in [0,T]\}$ are consistent (almost surely). It then follows that there is a continuous local martingale that starts from zero (with localizing sequence $\{\tau_m\}$), which we denote by $\{\int_0^t Y_s\,dW_s, t \in [0,T]\}$, that P-a.s. satisfies

$$\int_0^t Y_s\,dW_s = \lim_{m\to\infty} \int_0^t Y_s^{(m)}\,dW_s \quad \text{for all } \ t \in [0,T].$$

This limit process is unique up to indistinguishability.

Now suppose that W is an n-dimensional standard Brownian motion W defined on the complete probability space $(\Omega, \mathcal{F}, P)$ and $\{\mathcal{F}_t, t \in [0,T]\}$ is a filtration such that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ satisfies the usual conditions and $\{W_t, \mathcal{F}_t, t \in [0,T]\}$ is a continuous, n-dimensional martingale. (Again we could use the standard filtration associated with W for $\{\mathcal{F}_t\}$, but we do not assume that a priori.) Assume without loss of generality that $\mathcal{F} = \mathcal{F}_T$. Define $\mathcal{L}$ and $\mathcal{L}_{loc}$ in an analogous manner to that described above using this filtration. Suppose that $Y = (Y^1, \ldots, Y^n)$ is an n-dimensional process such that for each $i = 1, \ldots, n$, $Y^i \in \mathcal{L}_{loc}$ for each i. Then each of the one-dimensional stochastic integral processes $\{\int_0^t Y_s^i \, dW_s^i, t \in [0,T]\}$ can be defined as above, and then we define

$$\int_0^t Y_s \cdot dW_s = \sum_{i=1}^n \int_0^t Y_s^i \, dW_s^i, \quad t \in [0,T].$$

It can be shown using the independence of the components of W and the isometry for stochastic integrals that if $Y_i \in \mathcal{L}$ for all i, then

$$E\left[\left(\int_0^t Y_s \cdot dW_s\right)^2\right] = E\left[\int_0^t |Y_s|^2 \, ds\right], \quad t \in [0,T],$$

where $|Y_s| = \left(\sum\limits_{i=1}^n (Y_s^i)^2\right)^{\frac{1}{2}}$ for all $s \in [0,T]$.

D.3. Itô Process

Suppose that W is an n-dimensional Brownian motion (for some $n \geq 1$) defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ satisfying the usual conditions such that $\{W_t, \mathcal{F}_t, t \in [0,T]\}$ is a continuous n-dimensional martingale. Assume without loss of generality that $\mathcal{F} = \mathcal{F}_T$.

An *Itô process* driven by the Brownian motion W is a d-dimensional (for some $d \geq 1$), continuous, adapted process $X = \{X_t, t \in [0,T]\}$ satisfying P-a.s.,

$$X_t = X_0 + \int_0^t R_s \, ds + \int_0^t Z_s \cdot dW_s \quad \text{for all } t \in [0,T], \tag{D.1}$$

where

$$\left(\int_0^t Z_s \cdot dW_s\right)^i = \sum_{j=1}^n \int_0^t Z_s^{ij} \, dW_s^j \quad \text{for } i = 1, \ldots, d, \tag{D.2}$$

and

(i) $X_0 \in \mathcal{F}_0$;

(ii) $R = \{R_t, t \in [0,T]\}$ is a d-dimensional, adapted process such that $R : [0,T] \times \Omega \to \mathbb{R}^d$ is $(\mathcal{B}_T \times \mathcal{F}_T)$-measurable where $R(t,\omega) = R_t(\omega)$ for each $t \in [0,T]$ and $\omega \in \Omega$, and $\int_0^T |R_s|\, ds < \infty$ P-a.s. where $|R_s| = \left(\sum_{i=1}^d (R_s^i)^2\right)^{\frac{1}{2}}$ for $s \in [0,T]$;

(iii) $Z = \{Z_t, t \in [0,T]\}$ is an adapted process taking values in the set of $d \times n$ matrices with real entries (denoted by $\mathbb{R}^{d\times n}$), $Z : [0,T] \times \Omega \to \mathbb{R}^{d\times n}$ is $(\mathcal{B}_T \times \mathcal{F}_T)$-measurable where $Z(t,\omega) = Z_t(\omega)$ for each $t \in [0,T]$ and $\omega \in \Omega$, and $\int_0^T |Z_s|^2\, ds < \infty$ P-a.s. where

$$|Z_s| = \left(\sum_{i=1}^d \sum_{j=1}^n (Z_s^{ij})^2\right)^{\frac{1}{2}} \quad \text{for } s \in [0,T].$$

Remark. Given the above assumptions on R, P-a.s., the integral $\int_0^t R_s\, ds$ is well defined pathwise as a continuous function of $t \in [0,T]$ taking values in $\mathbb{R}^d$. For convenience, on an exceptional null set, we define the integrals for all $t \in [0,T]$ to be zero. This defines a continuous, adapted process. The adaptedness can be seen using the fact (cf. Chung and Williams [**9**], Theorem 3.7, Lemma 3.5(ii)) that there is a d-dimensional predictable process U such that

$$(\lambda \times P)(\{(t,\omega) \in [0,T] \times \Omega : R_t(\omega) \neq U_t(\omega)\}) = 0. \tag{D.3}$$

(A d-dimensional process $U = \{U_t, t \in [0,T]\}$ is *predictable* if $U : [0,T]\times\Omega \to \mathbb{R}^d$ given by $U(t,\omega) = U_t(\omega)$ for $t \in [0,T]$ and $\omega \in \Omega$, is measurable with the predictable σ-algebra on $[0,T] \times \Omega$ and the Borel σ-algebra on $\mathbb{R}^d$.) Using a suitable definition on an exceptional P-null set, $\{\int_0^t U_s\, ds, t \in [0,T]\}$ defines a continuous, adapted process. By (D.3), this process is indistinguishable from $\{\int_0^t R_s\, ds, t \in [0,T]\}$, and so (since the filtration satisfies the usual conditions) the latter is also an adapted process (it was already defined so as to be continuous). The integrals in (D.2) with respect to W are stochastic integrals as defined in Section D.2. By the assumptions on Z, these define continuous local martingales.

Sometimes for brevity we shall write the stochastic differential equation satisfied by an Itô process X in differential form:

$$dX_t = R_t\, dt + Z_t \cdot dW_t, \tag{D.4}$$

where the rigorous interpretation of this is as (D.1).

For fixed $i,j \in \{1,\dots,d\}$, the *mutual variation process* $\langle X^i, X^j\rangle$ associated with the i^{th} and j^{th} components of the Itô process X is given P-a.s., by

$$\langle X^i, X^j\rangle_t = \int_0^t (ZZ')_s^{ij}\, ds = \sum_{k=1}^n \int_0^t Z_s^{ik} Z_s^{jk}\, ds, \quad t \in [0,T], \tag{D.5}$$

where $'$ denotes transpose. When $i = j$, this yields the *quadratic variation process* associated with the component X^i. This process is sometimes also denoted by $[X^i]$, and P-a.s. is given by

$$[X^i]_t = \sum_{k=1}^{n} \int_0^t (Z_s^{ik})^2 \, ds, \quad t \in [0, T]. \tag{D.6}$$

D.4. Itô Formula

The following result is known as the *Itô formula.* For a proof of results that imply this, see Chung and Williams [**11**], Theorem 5.10. Let X be a d-dimensional Itô process as described in Section D.3 above. Suppose that D is a domain (i.e., an open connected set) in $\mathbb{R}^d$ such that

$$P(X_t \in D \text{ for all } t \in [0, T]) = 1.$$

Let $f : [0, T] \times D \to \mathbb{R}$ be a continuous function such that the partial derivatives $\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x_i}, \frac{\partial^2 f}{\partial x_i \partial x_j}$ exist and are continuous on $[0, T] \times D$. Then P-a.s. for all $t \in [0, T]$,

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \frac{\partial f}{\partial s}(s, X_s) \, ds + \sum_{i=1}^{d} \int_0^t \frac{\partial f}{\partial x_i}(s, X_s) \, R_s^i \, ds \\ &\quad + \sum_{i=1}^{d} \sum_{j=1}^{n} \int_0^t \frac{\partial f}{\partial x_i}(s, X_s) \, Z_s^{ij} \, dW_s^j \\ &\quad + \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(s, X_s) (ZZ')_s^{ij} \, ds. \end{aligned}$$

Here the integrals with respect to dW are stochastic integrals and those with respect to ds are defined a.s. as Lebesgue integrals. Using the notation (D.4) and (D.5), it is often convenient to write the above as

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \frac{\partial f}{\partial s}(s, X_s) \, ds + \sum_{i=1}^{d} \int_0^t \frac{\partial f}{\partial x_i}(s, X_s) \, dX_s^i \\ &\quad + \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(s, X_s) \, d\langle X^i, X^j \rangle_s, \end{aligned}$$

or in differential form

$$\begin{aligned} df(t, X_t) &= \frac{\partial f}{\partial t}(t, X_t) \, dt + \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(t, X_t) \, dX_t^i \\ &\quad + \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x_i \partial x_j}(t, X_t) \, d\langle X^i, X^j \rangle_t. \end{aligned}$$

In the simple case when $d = 1$ and $f : \mathbb{R} \to D$ is twice continuously differentiable, the Itô formula reduces to

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_s)\, R_s\, ds + \int_0^t f'(X_s)\, Z_s \cdot dW_s \\ &\quad + \frac{1}{2} \int_0^t f''(X_s)\, |Z_s|^2\, ds. \end{aligned}$$

D.5. Girsanov Transformation

The following result is frequently referred to as the Girsanov transformation or Girsanov theorem. The prototype of this formula for one-dimensional Brownian motion was developed by Cameron and Martin [**6, 7, 8**]. Subsequently, Girsanov [**18**] generalized their transformation to a multi-dimensional Brownian motion and used the modern notation and terminology of Itô's theory of stochastic integration. For a proof of this result, see Chung and Williams [**11**], Section 9.4; Karatzas and Shreve [**27**], Section 3.5; or Revuz and Yor [**35**], Chapter VIII.

Suppose that W is an n-dimensional Brownian motion (for some $n \geq 1$) defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ satisfying the usual conditions such that $\{W_t, \mathcal{F}_t, t \in [0, T]\}$ is a continuous n-dimensional martingale. Assume without loss of generality that $\mathcal{F} = \mathcal{F}_T$. Let $\kappa = \{\kappa_t, t \in [0, T]\}$ be an n-dimensional, adapted process such that $\kappa : [0, T] \times \Omega \to \mathbb{R}^n$ is $(\mathcal{B}_T \times \mathcal{F}_T)$-measurable where $\kappa(t, \omega) = \kappa_t(\omega)$ for all $t \in [0, T]$ and $\omega \in \Omega$, and suppose that $\int_0^T |\kappa_t|^2\, dt < \infty$ P-a.s. Then the stochastic integral process

$$\int_0^t \kappa_s \cdot dW_s = \sum_{i=1}^n \int_0^t \kappa_s^i\, dW_s^i, \quad t \in [0, T],$$

is well defined as a continuous local martingale. The integrals $\int_0^t \kappa_s\, ds$ and $\int_0^t |\kappa_s|^2\, ds$ are well defined for all $t \in [0, T]$, P-a.s. As explained in the remark in Section D.3, by defining these integrals to be zero for all $t \in [0, T]$ on an exceptional P-null set, we may assume that they define continuous adapted processes. Define

$$\Lambda_t = \exp\left(\int_0^t \kappa_s \cdot dW_s - \frac{1}{2} \int_0^t |\kappa_s|^2\, ds \right), \quad t \in [0, T].$$

Then, one can show using Itô's formula that $\Lambda = \{\Lambda_t, t \in [0, T]\}$ is a continuous local martingale satisfying P-a.s.,

$$\Lambda_t = 1 + \int_0^t \Lambda_s\, \kappa_s \cdot dW_s, \quad t \in [0, T].$$

Define

$$\tilde{W}_t = W_t - \int_0^t \kappa_s \, ds, \quad t \in [0,T].$$

For the following theorem, recall that we are assuming that $\mathcal{F} = \mathcal{F}_T$.

Theorem D.5.1. *(Girsanov's theorem) Suppose that $\{\Lambda_t, t \in [0,T]\}$ is a martingale. Define a new probability measure Q on $(\Omega, \mathcal{F})$ by*

$$Q(A) = E^P[1_A \Lambda_T] \quad \textit{for all} \quad A \in \mathcal{F}.$$

Then $\tilde{W}$ is a standard n-dimensional Brownian motion under Q.

Remark. In fact, Λ will be a martingale if

$$E^P[\Lambda_T] = 1.$$

A sufficient condition for this is what is known as *Novikov's criterion:*

$$E^P\left[\exp\left(\frac{1}{2}\int_0^T |\kappa_t|^2 \, dt\right)\right] < \infty.$$

For a proof of this sufficiency, see Karatzas and Shreve [**27**], Corollary 3.5.13, or Revuz and Yor [**35**], Proposition VIII.1.15.

D.6. Martingale Representation Theorem

Suppose that $W = \{W_t, t \in [0,T]\}$ defined on the complete probability space $(\Omega, \mathcal{F}, P)$ is a standard n-dimensional Brownian motion (for some $n \geq 1$) and let $\{\mathcal{F}_t, t \in [0,T]\}$ be its standard filtration. Without loss of generality we assume that $\mathcal{F} = \mathcal{F}_T$. The following result is used several times in the chapters on continuous models. It is important for this result that the filtration is the standard one generated by the Brownian motion. For a proof of this theorem, see for example Revuz and Yor [**35**], Theorem V.3.4.

Theorem D.6.1. *(Martingale Representation Theorem) Suppose that $\{M_t, \mathcal{F}_t, t \in [0,T]\}$ is a right continuous local martingale. There is an adapted, n-dimensional process $\eta = \{\eta_t, t \in [0,T]\}$ satisfying*

(i) *$\eta : [0,T] \times \Omega \to \mathbb{R}^n$ is $(\mathcal{B}_T \times \mathcal{F}_T)$-measurable where $\eta(t,\omega) = \eta_t(\omega)$ for $t \in [0,T]$ and $\omega \in \Omega$,*

(ii) *$\int_0^T |\eta_s|^2 \, ds < \infty$ P-a.s.,*

such that P-a.s.,

$$M_t = M_0 + \int_0^t \eta_s \cdot dW_s \quad \textit{for} \quad t \in [0,T].$$

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Index

自三十多年前 Black 和 Scholes 的开创性工作出现以来，金融数学这个现代学科无论在理论还是实践方面都经历了巨大发展。本书旨在介绍部分基础理论，使得学生和研究人员了解后能够阅读更高级的教科书和研究文章。

本书一开始讨论了欧式和美式衍生产品在离散二叉树模型（即离散时间和离散状态）下套期保值和定价的基本思想的发展，然后介绍了一个一般的离散有限市场模型，并在此场合中证明了资产定价的一些基本定理。概率论中的诸如条件期望、滤波、（超）鞅、等价鞅测度、鞅表示等工具，在这个简单的离散框架下被首次用到，从而搭建了通向连续（时间和状态）场合的桥梁，后者需要布朗运动和随机分析的概念。连续场合中最简单的模型是著名的 Black-Scholes 模型，欧式和美式衍生产品的定价和套期保值因此有所发展。本书最后介绍了连续市场模型的一些基本定理，这个模型在多个方面推广了简单 Black-Scholes 模型。

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